Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Geometry and Physics 56 (2006) 418–449

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

New integrable hierarchies from vertex operator representations of polynomial Lie algebras

Paolo Casati, Giovanni Ortenzi*

Dipartimento di Matematica e applicazioni, II Universitá di Milano Bicocca—Ed. U5, Via R. Cozzi 53, I-20126 Milan, Italy

Received 22 November 2004; accepted 24 February 2005
Available online 25 March 2005

Abstract

We give a representation–theoretic interpretation of recent discovered coupled soliton equations using vertex operators construction of affinization of not simple but quadratic Lie algebras. In this setup we are able to obtain new integrable hierarchies coupled to each Drinfeld–Sokolov of *A*, *B*, *C*, *D* hierarchies and to construct their soliton solutions.

AMS Classification: Primary 17B69; 37K10; Secondary 37K30

PACS: 02.30Ik; 05.45Yv

JGP SC: Classical integrable systems; Integrable PDEs; Vertex algebras

Keywords: Integrable hierarchies; Vertex operators; Soliton equations; Quadratic Lie algebras

1. Introduction

One of the most important achievement of the representations of the affine Lie algebras and their groups is surely the Lie theoretical explanation of the Hirota [7] bilinear approach to the soliton equations. This beautiful piece of mathematics is the result of an important sequence of relevant papers, which starts in 1981 with the works of Sato [21,23], where the

* Corresponding author. Tel.: +39 02 64485765; fax: +39 02 64485705.
E-mail addresses: casati@matapp.unimib.it (P. Casati), giolev@matapp.unimib.it (G. Ortenzi).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2005.02.010

link between the soliton equations and the infinite dimensional groups has been brought to the light for the first time. Some years later Date, Jimbo, Kashiwara and Miwa [3] gave a construction of the Kadomtsev–Petviashvili (KP) and Korteweg–de Vries (KdV) hierarchies in terms of the vertex operators representating the affine lie algebra $\mathfrak{a}_\infty$ and $\widehat{\mathfrak{sl}}_2$, respectively, while Segal and Wilson [24] have examined the same equation from a geometrical point of view. Finally Drinfeld, Sokolov [4], Kac Peterson and Wakimoto [10,13,12] have extended this theory to all affine Lie algebras. These results have suggested to find a similar interpretation for others hierarchies of soliton equations, for example in a recent work [2] Billig has obtained this goal for the sine-Gordon.

The main aim of the present paper is to contribute to the research in this direction. Our starting point is the following "coupled KdV equations" which appears in many very recent papers of different authors like Hirota et al. [8], Sakovich [20], Kakei [14]:

$$\begin{aligned} &v_t + 6vv_x + v_{xxx} = 0 \\ &w_t + 6vw_x + w_{xxx} = 0. \end{aligned} \tag{1.1}$$

The corresponding bilinear Hirota form of these equations (and actually of many others closely related, among them "coupled KP equations" [20]) is namely known [8] together with some soliton solutions, but, as far as we know, it is still missing their broader Lie-theoretic interpretation. In this paper we shall show how these equations are a particular case of a very wide class of "coupled soliton equations" which can be obtained using the vertex operator realization of a new class of infinite dimensional Lie algebras. These latter algebras are the affinization of not simple finite dimensional Lie algebras, which still posses a symmetric non degenerated ad-invariant bilinear form. Therefore, in our long journey towards our task we shall be enforced to develop a vertex operator algebras theory for a class of Lie algebras which are not the affinization of semisimple ones. Nevertheless our construction will allow us to produce coupled soliton equations corresponding to each of the Drinfeld–Sokolov and the AKP BKP CKP DKP hierarchies, although for sake of brevity only the case of the coupled AKP BKP and their reductions to opportune generalizations of the affine Lie algebras $A_1^{(1)}$ $A_2^{(1)}$, $A_1^{(2)}$ and $B_2^{(1)}$ are explicitly examinated in the paper. These reductions in turn provide a clear explanation of how the coupled KP equations become the coupled KdV ones by neglecting the dependency from one particular variable. Finally, the action on the space of representation of the corresponding infinite dimensional groups will provide, exactly as in the usual case, a class of multi-soliton solutions.

The paper is organized as follows: in the second section we shall describe a class of finite dimensional Lie algebras known in the literature as polynomial Lie algebras [15,19] which, roughly speaking, can be regarded as direct sum of semisimple Lie algebras endowed with a non canonical Lie bracket. We shall show that these Lie algebras can be constructed in completely different ways: namely as particular finite dimensional quotients of an infinite dimensional algebra and as a Wigner contraction of a direct sum of finite dimensional semisimple Lie algebras, or finally as tensor product between a finite dimensional Lie algebra $\mathfrak{g}$ and a nilpotent commutative ring. Further we shall show how on these Lie algebras is defined a class of symmetric non degenerated ad-invariant bilinear forms if a such bilinear form exits on $\mathfrak{g}$. In the next section it will be shown how these latter bilinear forms can be used to affinize those in general non simple Lie algebras. Then in Section 4 their vertex operator algebras construction is presented. Once this result is achieved we can tackle the problem to construct the corresponding generalized Hirota bilinear equation and their multisoliton solution in term of τ-functions. This will be done in the fifth and last section where further the case of the coupled AKP BKP and their reduction to Lie algebras generalizing the algebras $A_1^{(1)}$, $A_2^{(1)}$, $A_1^{(2)}$, and $B_2^{(1)}$ are presented into details.

2. The polynomial Lie algebras

The aim of this first section is to present a class of finite dimensional Lie algebras (called in what follows polynomial Lie algebras to keep the name usually used in the literature, see for example [15] and [19]) which are going to play a crucial role in the whole paper.

Definition 2.1. For any integer number n let $\mathfrak{g}^{(n)}$ denote the Lie algebra given by the tensor product

$$\mathfrak{g} \otimes \mathbb{C}^{(n)} \tag{2.1}$$

between the Lie algebra $\mathfrak{g}$ and the commutative ring $\mathbb{C}^{(n)} = \mathbb{C}[\lambda]/(\lambda)^{n+1}$.

This algebra may be identified with the Lie algebra of polynomial maps from $\mathbb{C}^{(n)}$ in $\mathfrak{g}$, hence an element $X(\lambda)$ in $\mathfrak{g}(\lambda)$ can be viewed as the mapping $X : \mathbb{C} \to \mathfrak{g}$, $X(\lambda) = \sum_{k=0}^{n} X_k\lambda^k$, where $X_k \in \mathfrak{g}$. In this setting the Lie bracket of two elements in $\mathfrak{g}(\lambda)$, $X(\lambda) = \sum_{k=0}^{n} X_k\lambda^k$ and $Y(\lambda) = \sum_{k=0}^{n} Y_k\lambda^k$ can be written explicitly as

$$[X(\lambda), Y(\lambda)] = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} [X_j, Y_{k-j}]_{\mathfrak{g}} \right) \lambda^k \tag{2.2}$$

where $[\cdot, \cdot]_{\mathfrak{g}}$ is the Lie bracket defined on $\mathfrak{g}$.

Observe that if $n > 0$ then $\mathfrak{g}^{(n)}$ fails to be semisimple, because $\mathfrak{g} \otimes \lambda^n$ is a non trivial ideal. Nevertheless if $\mathfrak{g}$ admits on it a symmetric ad-invariant non-degenerated bilinear form (i.e., if $\mathfrak{g}$ is a quadratic algebra) then roughly speaking this bilinear form is inherited by the whole Lie algebra $\mathfrak{g}^{(n)}$ it holds indeed:

Theorem 2.2. *Suppose that on $\mathfrak{g}$ is defined symmetric ad-invariant non-degenerated bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ then for any set of complex numbers $\mathcal{A} = \{a_j\}_{j=0,\ldots,n}$, the bilinear form*

$$\begin{aligned} &\mathfrak{g}^{(n)} \times \mathfrak{g}^{(n)} \to \mathbb{C} \\ &(X(\lambda), Y(\lambda)) \mapsto \langle X(\lambda), Y(\lambda) \rangle_{\mathcal{A}}^{(n)} = \textstyle\sum_{j=0}^{n} a_j \sum_{i=0}^{j} \langle X_i, Y_{j-i} \rangle_{\mathfrak{g}}. \end{aligned} \tag{2.3}$$

is a symmetric bilinear, ad-invariant and, if $a_n \neq 0$, not degenerate form.

Proof. The fact that (2.3) is a bilinear symmetric form follows immediately from the definition. Let us therefore first prove that the bilinear form (2.3) is ad-invariant. We have to prove that for any choice of elements $X(\lambda) = \sum_{k=0}^{n} X_k\lambda^k$, $Y(\lambda) = \sum_{j=0}^{n} Y_j\lambda^j$, $Z(\lambda) = \sum_{i=0}^{n} Z_i\lambda^i$ in $\mathfrak{g}^{(n)}$ it holds that

$$\langle [X(\lambda), Y(\lambda)], Z(\lambda) \rangle_{\mathcal{A}}^{(n)} = \langle X(\lambda), [Y(\lambda), Z(\lambda)] \rangle_{\mathcal{A}}^{(n)}. \tag{2.4}$$

Now

$$\begin{aligned}\langle [X(\lambda), Y(\lambda)], Z(\lambda)\rangle_{\mathcal{A}}^{(n)} &= \sum_{j=0}^{n} a_j \sum_{k=0}^{j} \langle [X, Y]_k, Z_{j-k}\rangle_{\mathfrak{g}} \\ &= \sum_{j=0}^{n} a_j \sum_{k=0}^{j} \left\langle \sum_{l=0}^{k} [X_l, Y_{k-l}], Z_{j-k} \right\rangle_{\mathfrak{g}} \end{aligned} \tag{2.5}$$

while

$$\begin{aligned}\langle X(\lambda), [Y(\lambda), Z(\lambda)]\rangle_{\mathcal{A}}^{(n)} &= \sum_{j=0}^{n} a_j \sum_{k=0}^{j} \langle X_k, [Y, Z]_{j-k}\rangle_{\mathfrak{g}} \\ &= \sum_{j=0}^{n} a_j \sum_{k=0}^{j} \left\langle X_k, \sum_{l=0}^{j-k} [Y_l, Z_{j-k-l}] \right\rangle_{\mathfrak{g}}. \end{aligned} \tag{2.6}$$

To see that (2.4) holds if suffices to observe that both (2.5) and (2.6) can be written as

$$\begin{aligned}\langle [X(\lambda), Y(\lambda)], Z(\lambda)\rangle_{\mathcal{A}}^{(n)} &= \textstyle\sum_{j=0}^{n} a_j \sum_{l_1+l_2+l_3=j} \langle [X_{l_1}, Y_{l_2}], Z_{l_3}\rangle_{\mathfrak{g}} \\ \langle X(\lambda), [Y(\lambda), Z(\lambda)]\rangle_{\mathcal{A}}^{(n)} &= \textstyle\sum_{j=0}^{n} a_j \sum_{l_1+l_2+l_3=j} \langle X_{l_1}, [Y_{l_2}, Z_{l_3}]\rangle_{\mathfrak{g}}\end{aligned}$$

and that the Eq. (2.4) immediately follows using the ad-invariance of the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. It remains to show that it is non-degenerated. We have to check that if $X(\lambda) \in \mathfrak{g}^{(n)}$ is such that $\langle X(\lambda), Y(\lambda)\rangle_{\mathcal{A}}^{(n)} = 0$ for every $Y(\lambda) \in \mathfrak{g}^{(n)}$ then $X(\lambda) = 0$. Indeed since $X(\lambda)$ has the form $X(\lambda) = \sum_{j=0}^{n} X_j \lambda^j$ its inner product with an element of the form $Y_n(\lambda) = Y_n \lambda^n$ will be:

$$\langle X(\lambda), Y_n(\lambda)\rangle_{\mathcal{A}}^{(n)} = a_n \langle X_0, Y_n\rangle_{\mathfrak{g}} \tag{2.7}$$

and since Y_n can be chosen arbitrarily in $\mathfrak{g}$, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is non degenerated on it and $a_n \neq 0$, $\langle X(\lambda), Y_n(\lambda)\rangle_{\mathcal{A}}^{(n)} = 0$ implies that $X_0 = 0$. Then by pairing $X(\lambda)$ with an element of the type $Y_{n-1}(\lambda) = Y_{n-1}\lambda^{n-1}$ we get:

$$\langle X(\lambda), Y_{n-1}(\lambda)\rangle_{\mathcal{A}}^{(n)} = a_n \langle X_1, Y_{n-1}\rangle_{\mathfrak{g}}$$

and then again $\langle X(\lambda), Y_{n-1}(\lambda)\rangle_{\mathcal{A}}^{(n)} = 0$ implies $X_1 = 0$. Repeating n times this argument we obtain step by step that each coefficient X_i is zero, proving the proposition. □

Remark 2.3. If we denote by ω the matrix representation of the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ defined on $\mathfrak{g}$ then it is immediately to show that the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{(n)}$ defined on $\mathfrak{g}^{(n)}$ has the matrix form:

$$\Omega_{\mathcal{A}}^{(n)} = \begin{pmatrix} a_0\omega & a_1\omega & a_2\omega & \cdots & a_{n-1}\omega & a_n\omega \\ a_1\omega & a_2\omega & & \cdots & a_n\omega & 0 \\ a_2\omega & & & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ a_{n-1}\omega & a_n\omega & 0 & \cdots & 0 & 0 \\ a_n\omega & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.8}$$

The just proved proposition shows that the polynomial Lie algebra $\mathfrak{g}^{(n)}$ when $\mathfrak{g}$ is semisimple, are non trivial (non abelian or semisimple) examples of quadratic Lie algebras i.e., finite dimensional Lie algebras which possess a symmetric ad-invariant, non degenerated bilinear form [16,17]. The very definition of the Lie algebra $\mathfrak{g}^{(n)}$ suggests a way to construct it in a form better suited for the purposes we have in mind. More precisely the following proposition allows us to obtain a matrix realization of $\mathfrak{g}^{(n)}$.

Proposition 2.4. *The map ρ given by*

$$\begin{aligned} &\rho : \mathbb{C}^{(n)}(\Lambda) \longrightarrow \mathrm{End}(\mathbb{C}^{(n+1)}) \\ &\rho(c_i \otimes \Lambda^i) \mapsto c_i\lambda^i \end{aligned} \tag{2.9}$$

where Λ is the $(n+1)\times(n+1)$ matrix given by

$$\Lambda = \sum_{i=0}^{n} e_{i+1,i} \tag{2.10}$$

and

$$(e_{ij})_{kr} = \begin{cases} 1 & if\, i=j,\ k=r \\ 0 & otherwise \end{cases}$$

is a ring homomorphism.

Now using together the definition of $\mathfrak{g}^{(n)}$ algebra and Proposition 2.4 we get a matrix representation of $\mathfrak{g}^{(k)}$.

Theorem 2.5. *If $\Pi : \mathfrak{g} \longrightarrow \mathrm{Aut}(\mathbb{C}^m)$ for some m is a true representation of $\mathfrak{g}$ then the map*

$$\tilde{\Pi} : \mathfrak{g}^{(n)} \mapsto \mathrm{Aut}(\mathbb{C}^{m(n+1)})$$

given by

$$\tilde{\Pi}(X_0, \ldots, X_n) = \sum_{i=0}^{n} X_i\Lambda^i = \begin{pmatrix} \Pi(X_0) & 0 & 0 & 0 & 0 & 0 \\ \Pi(X_1) & \Pi(X_0) & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \Pi(X_{n-1}) & \vdots & \cdots & \cdots & \Pi(X_0) & 0 \\ \Pi(X_n) & \Pi(X_{n-1}) & \cdots & \cdots & \Pi(X_1) & \Pi(X_0) \end{pmatrix} \tag{2.11}$$

is a true representation of $\mathfrak{g}^{(n)}$.

Proof. Since we have constructed a representation of $\mathbb{C}(\lambda^n)$ we have a representation of $\mathfrak{g} \otimes \mathbb{C}(\lambda^n)$ given by:

$$\Pi \otimes \rho : \mathfrak{g} \otimes \mathbb{C}(\lambda^n) \longrightarrow \mathrm{End}(\mathbb{C}^m) \otimes \mathrm{End}(\mathbb{C}^{n+1}) \cong \mathrm{End}(\mathbb{C}^{m(n+1)})$$

To bring it on $\mathfrak{g}^{(n)}$ directly we have only to use the isomorphism $\Phi : \mathfrak{g}^{(k)} \cong \mathfrak{g} \otimes \mathbb{C}(\lambda^k)$:

$$\tilde{\Pi} = \Pi \circ \rho \circ \Phi : \mathfrak{g}^{(k)} \longrightarrow \mathrm{Aut}(\mathbb{C}^{m(n+1)}) \quad \square$$

Previously in this section we have shown that if $\mathfrak{g}$ possesses an ad-invariant bilinear non degenerate form this gives rise a ad-invariant bilinear non degenerate form on $\mathfrak{g}^{(n)}$. It is therefore natural to wonder if this latter form has a natural expression in our matrix representation. This is actually the case. We have indeed for instance when $a_k = 1$ for all k that:

$$\langle (X_0, \ldots, X_n), (Y_0, \ldots, Y_n) \rangle^{(n)} = \mathrm{tr}(\tilde{\Pi}(X_0, \ldots, X_n)\tilde{\Pi}(Y_0, \ldots, Y_n)C^{(n)}) \tag{2.12}$$

where $C^{(n)}$ is the $m(n+1) \times m(n+1)$ matrix:

$$C^{(n)} = \begin{pmatrix} \frac{1}{n+1}\mathbb{I}_m & \frac{1}{n}\mathbb{I}_m & \cdots & \frac{1}{3}\mathbb{I}_m & \frac{1}{2}\mathbb{I}_m & \mathbb{I}_m \\ 0 & \frac{1}{n+1}\mathbb{I}_m & \frac{1}{n}\mathbb{I}_m & \cdots & \frac{1}{3}\mathbb{I}_m & \frac{1}{2}\mathbb{I}_m \\ 0 & 0 & \ddots & \ddots & \ddots & \frac{1}{3}\mathbb{I}_m \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \frac{1}{n+1}\mathbb{I}_m & \frac{1}{n}\mathbb{I}_m \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n+1}\mathbb{I}_m \end{pmatrix}$$

i.e. $C^{(n)}_{p,p+k} = \frac{1}{n+1-k}\mathbb{I}_m$, where $\mathbb{I}_m$ denotes the $m \times m$ identity matrix, $p = 0, \ldots, n-k$ and $k = 0, \ldots, n$ while $C^{(n)}_{pq} = 0$ if $q < p$.

3. The affine Lie algebras

In the previous section we have constructed a class of non semisimple Lie algebras which posses an ad-invariant non degenerate symmetric bilinear form. This their peculiar property suggests to investigate their affinization. Our construction will differ only in few details from that usually considered in the literature (see for example Kac [10]).

Let us consider a polynomial Lie algebra $\mathfrak{g}^{(n)}$ where $\mathfrak{g}$ is semisimple and let denote by $\mathcal{L}(\mathfrak{g}^{(n)})$ the corresponding loop algebra:

$$\mathcal{L}(\mathfrak{g}^{(n)}) = \mathfrak{g}^{(n)} \otimes_{\mathbb{C}} \mathbb{C}(t, t^{-1}) \tag{3.1}$$

where $\mathbb{C}(t, t^{-1})$ is the algebra of Laurent polynomials in a complex variable t. Remember that on it is defined an infinite complex Lie algebra bracket:

$$[X \otimes p, Y \otimes q] = [X, Y] \otimes pq \qquad (p, q \in \mathbb{C}(t, t^{-1}); X, Y \in \mathfrak{g}^{(n)}).$$

Then our "generalized affine Lie algebra" denoted by $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$ will be obtained by adding to $\mathcal{L}(\mathfrak{g}^{(n)})$ $n+1$ "central charges" and a "derivation" d. More precisely the Lie algebra $\mathcal{L}(\mathfrak{g}^{(n)})$ is the vector space

$$\hat{\mathcal{L}}(\mathfrak{g}^{(n)}) = \mathcal{L}(\mathfrak{g}^{(n)}) \oplus \sum_{i=0}^{n} \oplus \mathbb{C} c_i \oplus \mathbb{C} d \tag{3.2}$$

with Lie bracket defined as

$$\begin{aligned} &\left[(X_0, \ldots, X_n) \otimes t^p \oplus \left(\sum_{i=0}^{n} \nu_i c_i \oplus \mu d\right), (Y_0, \ldots, Y_n) \otimes t^q \oplus \left(\sum_{i=0}^{n} \nu_i^1 c_i \oplus \mu^1 d\right)\right] \\ &\quad = [(X_0, \ldots, X_n), (Y_0, \ldots, Y_n)] \otimes t^{p+q} \\ &\qquad + (\mu q (Y_0, \ldots, Y_n) \otimes t^q - \mu^1 p (X_0, \ldots, X_n) \otimes t^p) \\ &\qquad + p \delta_{p,-q} \sum_{i=0}^{n} \sum_{j=0}^{i} a_j \langle X_{i-j}, Y_j \rangle_{\mathfrak{g}} c_i. \end{aligned} \tag{3.3}$$

Observe that the element d acts as derivation on $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$ and that the Jacobi identity for this Lie bracket is granted from the ad-invariance of the bilinear form (2.3) on $\mathfrak{g}^{(n)}$, therefore the elements c_i $i = 1, \ldots, n$ play the role of $n+1$ linear independent central charges. Moreover it is a standard fact that this ad-invariant bilinear form can be extended to a symmetric ad-invariant bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{(n)t}$ on the whole Lie algebra $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$ by setting

$$\begin{aligned} &\langle X(t), Y(t) \rangle_{\mathcal{A}}^{(n)t} = \operatorname{Res}\left(\left\langle \frac{\mathrm{d}X(t)}{\mathrm{d}t}, Y(t) \right\rangle_{\mathcal{A}}^{(n)}\right) \quad \forall X(t), Y(t) \in \mathcal{L}(\mathfrak{g}^{(n)}) \\ &\langle d, d \rangle_{\mathcal{A}}^{(n)t} = 0 \\ &\langle c_i, d \rangle_{\mathcal{A}}^{(n)t} = 1 \quad \langle c_i, c_j \rangle_{\mathcal{A}}^{(n)t} = 0 \; i, j = 0, \ldots, n, \end{aligned} \tag{3.4}$$

where the Res is the linear functional of $\mathbb{C}(t, t^{-1})$ defined by the properties $\operatorname{Res}(t^{-1}) = 1$; $\operatorname{Res}(\frac{\mathrm{d}p}{\mathrm{d}t} = 0)$. For our purposes the most important case is when our finite dimensional Lie algebra $\mathfrak{g}$ is just a quadratic algebra but actually a complex semisimple Lie algebra. We want indeed in this case construct a vertex operator representation of the Lie algebra $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$. The first step in this direction [9,10] is to consider a Chevalley basis of $\mathfrak{g}$ adapted to the Cartan decomposition of our semisimple complex Lie algebra $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \oplus \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H) X \, \forall H \in \mathfrak{h}\} \tag{3.5}$$

where $\mathfrak{h}$ is an once for ever fixed Cartan subalgebra of $\mathfrak{g}$, and Δ is the corresponding the root system. More precisely if $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ is a set of simple roots, then a Chevalley basis is the set

$$\{H_{\alpha_1}, \ldots, H_{\alpha_r}\} \cup \{X_\alpha\}_{\alpha \in \Delta} \tag{3.6}$$

where the elements $\{H_{\alpha_1}, \dots, H_{\alpha_r}\}$ are the dual of the simple roots and therefore span $\mathfrak{h}$ and for any α in ΔX_α is a non trivial element X_α in g_α such that

$$H_{\alpha_i} = [X_{\alpha_i}, X_{-\alpha_i}] \quad [H, X_\alpha] = \alpha(H) X_\alpha \qquad \forall H \in \mathfrak{h}, i = 1, \dots, r.$$

This basis in turn allows us to define a basis for the whole Lie algebra $\mathfrak{g}^{(n)}$ and its affinization $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$. They will be namely respectively for $\mathfrak{g}^{(n)}$

$$\{H^k_{\alpha_1}, \dots, H^k_{\alpha_r}\} \cup \{X^k_\alpha\}_{\alpha\in\Delta} \qquad k = 0, \dots, n \tag{3.7}$$

and for $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$

$$\begin{cases} \{H^k_{\alpha_1} \otimes t^{m_1}, \dots, H^k_{\alpha_r} \otimes t^{m_r}\} \cup \{X^k_\alpha \otimes t^{m_\alpha}\}_{\alpha\in\Delta} \quad k = 0, \dots, n, m_i \in \mathbb{Z} \\ c_0, \dots, c_n \\ d. \end{cases} \tag{3.8}$$

The corresponding Lie bracket are for $\mathfrak{g}^{(n)}$

$$\begin{aligned} &[H^k_{\alpha_i}, H^j_{\alpha_s}] = 0 \\ &[H^k_{\alpha_i}, X^j_\alpha] = \begin{cases} \alpha(H^k_{\alpha_i}) X^{k+j}_\alpha & \text{if } j+k \le n \\ 0 & \text{otherwise} \end{cases} \\ &[X^k_\alpha, X^j_\beta] = \begin{cases} N(\alpha,\beta) X^{k+j}_{\alpha+\beta} & \text{if } j+k \le n \text{ and } \alpha+\beta \in \Delta \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.9}$$

with opportune integer numbers $N(\alpha, \beta)$. While for the Lie affine algebra $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$ they are

$$\begin{aligned} &[H^k_{\alpha_i} \otimes t^{m_i}, H^j_{\alpha_s} \otimes t^{m_s}] = \langle H^k_{\alpha_i}, H^j_{\alpha_s}\rangle^{(n)}_{\mathcal{A}} \delta_{-m_i, m_s} c_{k+j} \\ &[H^k_{\alpha_i} \otimes t^{m_i}, X^j_\alpha \otimes t^{m_\alpha}] = \begin{cases} \alpha(H_{\alpha_i}) X^{k+j}_\alpha \otimes t^{m_i+m_\alpha} & \text{if } j+k \le n \\ 0 & \text{otherwise} \end{cases} \\ &[X^k_\alpha \otimes t^{m_\alpha}, X^j_\beta \otimes t^{m_\beta}] = \begin{cases} N(\alpha,\beta) X^{k+j}_{\alpha+\beta} \otimes t^{m_\alpha+m_\beta} \\ \quad + \langle X^k_\alpha, X^j_\beta\rangle^{(n)}_{\mathcal{A}} \delta_{m_\alpha,-m_\beta} c_{j+k} & \text{if } j+k \le n \text{ and } \alpha + \beta \in \Delta \\ 0 & \text{otherwise} \end{cases} \\ &[d, X \otimes t^m] = m X^k \otimes t^m \qquad \forall X \in \mathfrak{g}^{(n)} \\ &[c_i, X] = 0 \qquad \forall X \in \hat{\mathcal{L}}(\mathfrak{g}^{(n)}), j = 0, \dots, n. \end{aligned} \tag{3.10}$$

The importance of this basis in what follows is due to the fact that it allows us to define some generating series through which our searched vertex operators are constructed. These generating series are defined by the formulas: ([10,6]):

$$\begin{aligned} &H^k_{\alpha_i}(z) = \textstyle\sum_{n\in\mathbb{Z}} H^k_{\alpha_i} \otimes t^n z^{-1-n} \\ &X^k_{\alpha_i}(z) = \textstyle\sum_{n\in\mathbb{Z}} X^k_{\alpha_i} \otimes t^n z^{-1-n} \\ &\partial_z = \tfrac{\mathrm{d}}{\mathrm{d}z} \end{aligned} \tag{3.11}$$

where z is a formal variable. Using these formal operators is immediate to prove

Lemma 3.1. *The Lie brackets (3.10) for the affine algebra $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$ are equivalent to*:

$$\begin{aligned}
&[H^k_{\alpha_i}(z_1), H^j_{\alpha_s}(z_2)] = \langle H^k_{\alpha_i}, H^j_{\alpha_s}\rangle^{(n)}_{\mathcal{A}}(\partial_{z_2}\delta)(z_1-z_2)c_{k+j}\\
&[H^k_{\alpha_i}(z_1), X^j_{\alpha}(z_2)] = \begin{cases} \alpha(H^k_{\alpha_i})X^{k+j}_{\alpha}(z_2)\delta(z_1-z_2) & k+j\le n\\ 0 & \text{otherwise}\end{cases}\\
&[X^k_{\alpha}(z_1), X^j_{\beta}(z_2)] = \begin{cases} N(\alpha,\beta)X^{k+j}_{\alpha+\beta}(z_2)\delta(z_1-z_2) - \langle X^k_{\alpha}, X^j_{\beta}\rangle^{(n)}_{\mathcal{A}}(\partial_{z_2}\delta)(z_1-z_2)c_{k+j} & k+j\le n\\ 0 & \text{otherwise}\end{cases}\\
&[d, X(z)] = -(z\partial_z+1)X(z) \qquad \forall X\in\mathfrak{g}^{(n)}\\
&[c_i, X(z)] = 0 \qquad \forall X(z)\in\hat{\mathcal{L}}(\mathfrak{g}^{(n)}), \quad i=0,\ldots,n
\end{aligned}$$

where $\delta(z_1-z_2) = z_1^{-1}\sum_{n\in\mathbb{Z}}\frac{z_1^n}{z_2^n}$.

Until now we did not impose any restriction on the subset of complex numbers $\{a_0,\ldots,a_n\}$, which appear in the definition of the bilinear form (2.3), but in view of their realization as vertex operator algebra on "generalized" Fock spaces we need to suppose that every a_k is different from zero which, without loss of generality boils down to set $a_k = 1$ for every k.

4. Vertex algebras representations

Now we can describe the construction of the vertex operators representation of our Lie algebras $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$ in the case when $\mathfrak{g}$ is a simple complex Lie algebra. Using the same constructions of the first section it can be easily proved that the Lie algebra $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$ is isomorph to the Lie algebra tensor product:

$$\hat{\mathcal{L}}(\mathfrak{g}^{(n)}) \simeq (\tilde{\mathcal{L}}(\mathfrak{g})\otimes\mathbb{C}^{(n)}(\lambda))\rtimes\mathbb{C}d \tag{4.1}$$

where $\tilde{\mathcal{L}}(\mathfrak{g})$ denotes as usual [10] the central extension of the loop algebra $\mathcal{L}(\mathfrak{g}) = \mathfrak{g}\otimes\mathbb{C}(t,t^{-1})$ (which is of course a one codimensional subalgebra of the affine Lie algebra $\hat{\mathcal{L}}(\mathfrak{g})$); and d acts as the derivation $t\frac{\mathrm{d}}{\mathrm{d}t}$ on $\mathbb{C}(t,t^{-1})$ while its action on the other factors is trivial. This equivalence suggests of course a way to obtain a generalized vertex operators representation of $\hat{\mathcal{L}}(\mathfrak{g}^{(n)})$, namely if $\Gamma:\tilde{\mathcal{L}}(\mathfrak{g})\to\mathrm{End}(V)$ is the restriction to $\tilde{\mathcal{L}}(\mathfrak{g})$ of a vertex operator representation for $\hat{\mathcal{L}}(\mathfrak{g})$ and $\rho:\mathbb{C}^{(n)}(\lambda)\to\mathrm{End}(\mathbb{C}^{n+1})$ is the representation (2.11) of $\mathbb{C}^{(n)}(\lambda)$ then our "vertex operators representations" will be up the derivation "the tensor product" of the two:

$$\begin{aligned}
&\Pi:\hat{\mathcal{L}}(\mathfrak{g}^{(n)}) \longrightarrow \mathrm{End}(V\otimes\mathbb{C}^{n+1})\\
&\Pi(X\otimes p(\lambda)) \mapsto \Gamma(X)\otimes\rho(p(\lambda))\\
&\Pi(d)\mapsto D\otimes 1.
\end{aligned}\tag{4.2}$$

where D coincides with the action of the derivation of $\hat{\mathcal{L}}(\mathfrak{g})$ on $\mathrm{End}(V)$. Indeed we have only to check that $[\Pi(X\otimes p(\lambda)), \Pi(D\otimes 1)] = \Pi([X\otimes p(\lambda), \Pi(D\otimes 1)])$ which follows immediately from $[\Gamma(X)\otimes\rho(p(\lambda)), D\otimes 1] = [\Gamma(X), D]\otimes\rho(p(\lambda))$. Since this is the main object of the present work let us explain into details this construction when the vertex representation Γ of $\hat{\mathcal{L}}(\mathfrak{g})$ is the basic homogeneous representation [10,11,6].

More precisely let Q be the root lattice associated with the simple Lie algebra $\mathfrak{g}$, which we suppose to be of rank l and let $\mathbb{C}(Q)$ be its group algebra, i.e., the algebra with basis $e^{\alpha}\alpha \in Q$ and multiplication:

$$e^{\alpha}e^{\beta} = e^{\alpha+\beta}, \quad e^0 = 1.$$

We shall denote by $\mathfrak{h} = Q \otimes_{\mathbb{Z}} \mathbb{C}$ the complexification of Q and by

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}(t, t^{-1}) \oplus \mathbb{C}$$

the affinization of $\mathfrak{h}$, and finally by S the symmetric algebra over the space $\mathfrak{h}^{<0} = \sum_{j<0} \mathfrak{h} \otimes t^j$ (following the literature we shall write Ht^j in place of $H \otimes t^j$). Then we can define a representation π of $\hat{\mathfrak{h}}$ on $V_Q = S \otimes \mathbb{C}(Q)$ by setting $\pi = \pi_1 \otimes \pi_2$ where π_1 acts on S as

$$\begin{aligned} &\pi_1(c_0) &&= I \\ &\pi_1(Ht^n)(At^s) &&= \begin{cases} HAt^{n-s} & \text{if } n < 0 \\ n\delta_{n,s}\langle H|A\rangle_{\mathfrak{g}} & \text{if } n \geq 0 \end{cases} \end{aligned} \tag{4.3}$$

while π_2 act on (Q) simply by

$$\pi_2(K) = 0, \qquad \pi_2(Ht^n)e^{\alpha} = \delta_{n,0}\langle\alpha|H\rangle_{\mathfrak{g}}e^{\alpha}. \tag{4.4}$$

Let now set $H_\alpha = \alpha^v \otimes 1$, where α^v is the dual element in $\mathfrak{h}$ of $\alpha\, \alpha(n) = \pi(H_\alpha t^n)$, $H_n = \pi(Ht^n)$, and e^{α} the operator on V_Q of multiplication by $1 \otimes e^{\alpha}$. Then let us consider the following $\mathrm{End}(V_Q)$-valued fields:

$$\begin{aligned} &H(z) = \textstyle\sum_{n\in\mathbb{Z}} H_n z^{-n-1} \\ &\Gamma_\alpha(z) = \textstyle\sum_{k=0}^{n} \left(\exp\left(\sum_{n\geq 1} \frac{\alpha(-n)z^n}{n}\right)\right)\left(\exp\left(\sum_{n\geq 1} \frac{\alpha(n)z^{-n}}{n}\right)\right) e^{\alpha}z^{\alpha}b_\alpha \end{aligned} \tag{4.5}$$

where b_α acts on V_Q as

$$b_\alpha(s \otimes e^{\alpha}) = \epsilon(\alpha, \beta)s \otimes e^{\alpha}$$

with $\epsilon : Q \to \{\pm\}$ is a two-cocycle ([11]) such that

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)+(\alpha|\alpha)(\beta|\beta)}.$$

Using these notations and Theorem 2.5 we can prove.

Theorem 4.1. *Let $V_Q^n = \oplus_{i=0}^{n} V_Q$ be the direct sum of $n+1$ copies of V_Q then the following* $\mathrm{End}(V_Q^N)$*-valued fields*:

$$\begin{aligned} &c_k = c\Lambda^k, \qquad k = 0, \ldots, n, \quad c \in \mathbb{C} \\ &H_\alpha^k(z) = \textstyle\sum_{m\in\mathbb{Z}} \alpha(m)z^{-m-1}\Lambda^k = H_\alpha(z)\Lambda^k, \quad k = 0, \ldots, n \\ &\Gamma_\alpha^k(z) = \textstyle\sum_{k=0}^{n} \left(\exp\left(\sum_{m\geq 1} \frac{\alpha(-m)z^m}{m}\right)\right)\left(\exp\left(\sum_{m\geq 1} \frac{\alpha(m)z^{-m}}{m}\right) e^{\alpha}z^{\alpha}\right)\Lambda^k \\ &\qquad = \Gamma_\alpha(z)\Lambda^k, \quad k = 0, \ldots, n \end{aligned} \tag{4.6}$$

and the last formula of (4.2) define a vertex operator representation of the Lie algebra $\hat{\mathcal{L}}(\mathfrak{g})\otimes\mathbb{C}^{(n)}(\lambda)$, *where the matrix* Λ *is given by Eq. (2.10).*

Proof. As usual we need only to check that our generating series satisfies the right OPE. But this can be easily done, keeping in mind the OPE of the fields (4.5) (see [10,11]). We have indeed:

$$\begin{aligned} H^k_\alpha(z)H^j_\beta(w) &= H_\alpha(z)H_\beta(w)\Lambda^k\Lambda^j = \begin{cases}\sim \frac{\langle H_\alpha|H_\beta\rangle}{(z-w)^2}c_{k+j} & \text{if } k+j\le n\\ 0 & \text{otherwise}\end{cases}\\ H^k(z)\Gamma^j_\alpha(w) &= H(z)\Gamma_\alpha(z)\Lambda^k\Lambda^j \quad = \begin{cases}\sim \frac{\langle H,\alpha\rangle}{z-w}\Gamma_{\alpha(z)}\Lambda^{k+j} & \text{if } k+j\le n\\ 0 & \text{otherwise.}\end{cases}\end{aligned}\tag{4.7}$$

In similar way

$$\begin{aligned}\Gamma^k_\alpha(z)\Gamma^j_\beta(w) &= \Gamma_\alpha(z)\Gamma_\beta(w)\Lambda^k\Lambda^j \quad = 0 && \text{if } \alpha+\beta\notin\Delta\\ \Gamma^k_\alpha(z)\Gamma^j_\beta(w) &= \Gamma_\alpha(z)\Gamma_\beta(w)\Lambda^k\Lambda^j \quad = \begin{cases}\sim\epsilon(\alpha,\beta)\frac{\Gamma_{\alpha+\beta}}{z-w}\Lambda^{k+j} & \text{if } k+j\le n\\ 0 & \text{otherwise;}\end{cases} && \text{if } \alpha+\beta\in\Delta\\ \Gamma^k_\alpha(z)\Gamma^j_{-\alpha}(w) &= \Gamma_\alpha(z)\Gamma_{-\alpha}(z)\Lambda^k\Lambda^j = \begin{cases}\sim\epsilon(\alpha,-\alpha)\frac{c_{k+j}}{(z-w)^2}+\frac{\alpha(w)}{z-w}\Lambda^{k+j} & \text{if } k+j\le n\\ 0 & \text{otherwise.}\end{cases}\end{aligned}\tag{4.8}$$

□

4.1. Generalized boson–fermion correspondence

In the next section we shall apply the theory of Kac Wakimoto [13] on order to obtain a class of coupled soliton equations. Although this theory may be implemented using directly the vertex operators given in Theorem 4.1 even in this case there exists a generalized fermionic construction which is in our opinion worth to be presented at least in the case in which the simple Lie algebra $\mathfrak{g}$ is of type A.

Let us first consider the direct sum of $n+1$ copies of infinite dimensional wedge algebras:

$$F^{(n)} = \oplus_{i=0}^n F^i \tag{4.9}$$

where the spaces F^i with $i = 0,\ldots,n$ are isomorph to the infinite wedge space F generated by the semi-infinite monomials

$$\underline{i}_1\wedge\underline{i}_2\wedge\cdots\wedge\underline{i}_j\wedge\cdots$$

where the i_j are integers such that

$$i_1 > i_2 > i_3 \quad \text{and} \quad i_j = i_{j-1}-1 \quad \text{for } n \text{ big enough}$$

(see [10] for more details). Every space F^i has a charge decomposition

$$F^i = \oplus_{m\in\mathbb{Z}} F^i_m$$

where F^i_m is the linear space spanned by all semi-infinite monomials in F^i which differ from the vector

$$|m\rangle^i = (\underline{m} \wedge \underline{m-1} \wedge \underline{m-2} \wedge \cdots)^i$$

called the vacuum vector of charge m in F^i only at finite number of places. Obviously the same decomposition exists for the whole space $F^{(n)}$ as $F^{(n)} = \oplus_{m\in\mathbb{Z}} F_m^{(n)}$ where $F_m^{(n)} = \oplus_{i=0}^n F_m^i$.

On End$(F^{(n)})$ we define the following operators $\psi_i^{(k)}$ and $\psi_i^{*(k)}$ ($k = 0, \ldots, n, i \in \mathbb{N}$):

$$\begin{aligned}
&\psi_i^{(k)}((\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^j, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^n)\\
&\quad = \textstyle\sum_{l=0}^{n-k} \psi_i e_{l+k,l}((\underline{i}_1 \wedge \underline{i}_2 \cdots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^j, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^n)\\
&\quad = (\underbrace{0, \ldots, 0}_k, (\psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^0)^k, \ldots, (\psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^j)^{j+k}, \ldots,\\
&\qquad (\psi_i(\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^{n-k})^n)\\
&\psi_i^{*(k)}((\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^j, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^n)\\
&\quad = \textstyle\sum_{l=0}^{n-k} \psi_i^* e_{l+k,l}((\underline{i}_1 \wedge \underline{i}_2 \cdots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^j, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^n)\\
&\quad = (0, \ldots, 0, (\psi_i^*(\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^0)^k, \ldots, (\psi_i^*(\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^j)^{j+k}, \ldots,\\
&\qquad (\psi_i^*(\underline{i}_1 \wedge \underline{i}_2 \wedge \ldots)^{n-k})^n)
\end{aligned} \tag{4.10}$$

where the action of the operators ψ_j and ψ_j^* is given by the formula [10]:

$$\begin{aligned}
&\psi_j((\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^m) = \begin{cases} 0 & \text{if } j = i_s \text{ for some } s\\ (-1)^s(\underline{i}_1 \wedge \cdots \wedge \underline{i}_s^m \wedge \underline{i}_{i_s+1} \wedge \cdots)^m & \text{if } i_s > j > i_{s+1} \end{cases}\\
&\psi_j^*((\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^m) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s\\ (-1)^{s+1}(\underline{i}_1 \wedge \cdots \wedge \underline{i}_{s-1} \wedge \underline{i}_{i_s+1} \wedge \cdots)^m & \text{if } j = i_s. \end{cases}
\end{aligned} \tag{4.11}$$

A simple computation shows that the operators defined above satisfies the equations

$$\begin{aligned}
&\psi_i^{(k)}\psi_l^{(j)} + \psi_l^{(j)}\psi_i^{(k)} = 0, \qquad \psi_i^{*(k)}\psi_l^{*(j)} + \psi_l^{*(j)}\psi_i^{*(k)} = 0\\
&\psi_i^{(k)}\psi_l^{*(j)} + \psi_l^{*(j)}\psi_i^{(k)} = \begin{cases} \delta_{il}\Lambda^{j+k} & \text{if } j + k \leq n\\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.12}$$

defining a generalized polynomial Clifford algebra of type A, which in what follows will be denoted by $CL^{(n)}$. This name may be justified observing that the same algebra can be constructed by performing a universal algebraic construction starting from an opportune infinite dimensional vector space endowed with a symmetric bilinear form related to the bilinear form (2.3) of $\mathfrak{g}^{(n)}$. It is clear that the vector

$$|0\rangle = ((0 \wedge -1 \wedge -2 \wedge \cdots)^0, \underbrace{0, \ldots, 0}_n)$$

is a cyclic vector with respect the action of $CL^{(n)}$ (i.e., $F^{(n)} = CL^{(n)}(|0\rangle)$) which satisfies the relations

$$\psi_j^{(k)}|0\rangle = 0 \quad \text{for } j \leq 0, \qquad \psi_j^{*(k)}|0\rangle = 0 \quad \text{for } j > 0, \qquad k = 0, \ldots, n.$$

Moreover it can be also checked that the operator $\psi_i^{*(k)}$ is the adjoint of the operator $\psi_i^{(k)}$ with respect to the bilinear (non degenerated but not positive definite) form on $F^{(n)}$ given by

$$\langle(\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^n | (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^0, \ldots, (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^n\rangle_{F^{(n)}} = \sum_{l=0}^{n}\sum_{m=0}^{l}\langle(\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^l | (\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots)^{l-m}\rangle \tag{4.13}$$

where $\langle\cdot|\cdot\rangle$ denote the Hermitian form on F for which the canonical basis for F is orthonormal. The more significant consequence of the equations (4.12) are the following commutation relations:

$$\begin{aligned}[]
[\psi_i^{(l)}\psi_j^{*(m-l)}, \psi_k^{(p)}] &= \begin{cases} \delta_{kj}\psi_i^{(m+p)} & \text{if } m+p \le n \\ 0 & \text{otherwise} \end{cases} \\
[\psi_i^{(l)}\psi_j^{*(m-l)}, \psi_k^{*(p)}] &= \begin{cases} -\delta_{ki}\psi_j^{*(m+p)} & \text{if } m+p \le n \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{4.14}$$

which can be checked as follows

$$\begin{aligned}[]
[\psi_i^{(l)}\psi_j^{*(m-l)}, \psi_k^{(p)}] &= \psi_i^{(l)}\psi_j^{*(m-l)}\psi_k^{(p)} - \psi_k^{(p)}\psi_i^{(l)}\psi_j^{*(m-l)} \overset{(4.12)}{=} \psi_i^{(l)}\psi_j^{*(m-l)}\psi_k^{(p)} \\
&\quad + \psi_i^{(l)}\psi_k^{(p)}\psi_j^{*(m-l)} \\
&= \psi_i^{(l)}(\psi_j^{*(m-l)}\psi_k^{(p)} + \psi_k^{(p)}\psi_j^{*(m-l)}) \overset{(4.12)}{=} \psi_i^{(l)}\delta_{jk}\Lambda^{m+p-l} \\
&= \begin{cases} \delta_{kj}\psi_i^{(m+p)} & \text{if } m+p \le n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

while the similar proof for the second one is left to the reader.

The importance of equations (4.14) is due to the fact that can be used to define a representation of a "polynomial" generalization of the infinite dimensional Lie algebra $\mathfrak{gl}_\infty$ i.e., using Definition 2.1 of the Lie algebra:

$$\mathfrak{gl}_\infty^{(n)} = \mathfrak{gl}_\infty \otimes \mathbb{C}^{(n)}(\lambda) \tag{4.15}$$

Therefore a moment's reflection shows that $\mathfrak{gl}_\infty^{(n)}$ is the Lie algebra given by the linear span of the basis $\{E_{ij}^k\}_{j,i\in\mathbb{Z},k=0,\ldots,n}$ with Lie brackets given by the formulas:

$$[E_{ij}^k, E_{lm}^s] = \begin{cases} \delta_{jl}E_{im}^{k+s} - \delta_{im}E_{lj}^{k+s} & \text{if } k+s \le n \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

Similarly, starting from equations (4.12), one can view the polynomial Clifford algebra $CL^{(n)}$ as the tensor product $CL \otimes \mathbb{C}^{(n)}(\lambda)$.

The representation of the Lie algebra $\mathfrak{gl}_\infty^{(n)}$ on $F^{(n)}$ is given as follows.

Theorem 4.2. *The map* $\Psi : \mathfrak{gl}_\infty^{(n)} \to \mathrm{End}(F^{(n)})$ *given by*:

$$\Psi(E_{ij}^k) = \frac{1}{k+1}\sum_{l=0}^{k}\psi_i^{(k-l)}\psi_j^{*(l)}, \qquad i, j \in \mathbb{Z}, \quad k = 0, \ldots, n \tag{4.17}$$

defines a representation Ψ of $\mathfrak{gl}_\infty^{(n)}$ on $F^{(n)}$.

Proof. Using formulas (4.14) we have for $k+s \le n$:

$$
\begin{aligned}
[\Psi(E_{ij}^k), \Psi(E_{lm}^s)] &= \frac{1}{k+1}\frac{1}{s+1}\left[\sum_{p=0}^{k} \psi_i^{(k-p)}\psi_j^{*(p)}, \sum_{q=0}^{s} \psi_l^{(s-q)}\psi_m^{*(q)}\right] \\
&= \frac{1}{k+1}\frac{1}{s+1}\sum_{p=0}^{k}\sum_{q=0}^{s}([\psi_i^{(k-p)}\psi_j^{*(p)}, \psi_l^{(s-q)}\psi_m^{*(q)}]) \\
&= \frac{1}{k+1}\frac{1}{s+1}\sum_{p=0}^{k}\sum_{q=0}^{s}([\psi_i^{(k-p)}\psi_j^{*(p)}, \psi_l^{(s-q)}]\psi_m^{*(q)}) \\
&\quad + \frac{1}{k+1}\frac{1}{s+1}\sum_{p=0}^{k}\sum_{q=0}^{s}(\psi_i^{(k-p)}[\psi_j^{*(p)}, \psi_l^{(s-q)}\psi_m^{*(q)}]) \\
&= \frac{1}{s+1}\frac{1}{k+1}\sum_{p=0}^{k}\sum_{q=0}^{s}(\delta_{jl}\psi_i^{(k+s-q)}\psi_m^{*(q)} - \delta_{mi}\psi_l^{(s-q)}\psi_j^{*(k+q)}) \\
&= \delta_{jl}\frac{1}{s+1}\sum_{q=0}^{s}\psi_i^{(k+s-q)}\psi_m^{*(q)} - \delta_{mi}\frac{1}{s+1}\sum_{q=0}^{k}(\psi_l^{(s-q)}\psi_j^{*(k+q)}) \\
&= \delta_{jl}\Psi(E_{im}^{k+s}) - \delta_{mi}\Psi(E_{lj}^{k+s}) = \Psi([E_{ij}^k, E_{lm}^s])
\end{aligned}
$$

since it is easily checked that $\frac{1}{s+1}\sum_{q=0}^{k}\psi_l^{(s-q)}\psi_j^{*(k+q)} = \Psi(E_{lj}^{k+s})$ and $\frac{1}{s+1}\sum_{q=0}^{s}\psi_i^{(k+s-q)}\psi_m^{*(q)} = \Psi(E_{im}^{k+s})$. While if $k+s>n$ a similar computation gives $[\Psi(E_{ij}^k), \Psi(E_{lm}^s)] = 0$ as wanted. □

Note that, while the action of $\mathrm{CL}^{(n)}$ interchanges the charges, every subspaces $F_m^{(n)}$ is left invariant by the representation Ψ. Further these latter spaces are indecomposable so that Ψ is the direct sum of its restrictions on $F_m^{(n)}$.

We are actually also interested in the corresponding group representation, despite the fact that $\mathfrak{gl}_\infty^{(n)} = \mathrm{Lie}(GL_\infty \ltimes \mathfrak{gl}_\infty)$ the exponential map of this algebra lies in a bigger group, which contains $(GL_\infty \ltimes \mathfrak{gl}_\infty)$ as a proper subgroup [18].

Proposition 4.3. *Let $N_\infty^{(n)}$ be the following subset of $\mathfrak{gl}_\infty^{(n)}$ regarded as associative ring*:

$$N_\infty^{(n)} = \{I + X | X \in \mathfrak{gl}_\infty \otimes \lambda\mathbb{C}^{(n-1)}(\lambda)\}$$

then

1. *$N_\infty^{(n)}$ is a group with respect the ring's product.*

2. *The group $G_\infty^{(n)} = GL_\infty \ltimes N_\infty^{(n)}$ is the littlest group which contains the image of the exponential map on $\mathfrak{gl}_\infty^{(n)}$.*

Proof.

1. Since it is obvious that the set $N_\infty^{(n)}$ is closed with respect to the ring's product it is enough to show that it contains the inverse of any its element. But this is simple a matter of computation we have indeed that
$$(I+X)\left(\sum_{k=0}^{n}(-1)X^k\right) = I$$
with $\sum_{k=0}^{n}(-1)X^k$ which belongs to $N_\infty^{(n)}$.
2. The second statement follows immediately once one recognizes that GL_∞ is the exponential group of the Lie algebra $\mathfrak{gl}_\infty$ [10] and that $\mathfrak{gl}_\infty^{(n)} = \mathfrak{gl}_\infty \ltimes \mathfrak{n}^{(n)}$ where $\mathfrak{n}^{(n)}$ is the Lie algebra $\mathfrak{gl}_\infty \otimes \lambda\mathbb{C}^{(n-1)}(\lambda)$. □

The representation Ψ can be exponentiated to the Lie group $G_\infty^{(n)}$. Namely if g is an element of $G_\infty^{(n)}$ of the form $g = \exp X_0$ where X_0 belongs to $\mathfrak{gl}_\infty \otimes 1 \simeq \mathfrak{gl}_\infty$ then we have the natural extension of the usual case [10]

$$\begin{aligned}&\Psi(g)((\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^0,\ldots,(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^j,\ldots,(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^n)\\ &=((g(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^0)^0,\ldots,(g(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^j)^j,\ldots,(g(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^n)^n)\end{aligned}\tag{4.18}$$

where

$$(g(\underline{i}_1\wedge\underline{i}_2\cdots)^s)^s = \left(\sum_{j_1>j_2>\cdots}\det(g^{i_1,i_2,\ldots}_{j_1,j_2,\ldots})(j_1\wedge j_2\cdots)\right)^s.$$

While if g is an element of $G_\infty^{(n)}$ of the form $g = \exp X_k$ with X_k which belongs to $\mathfrak{gl}_\infty \otimes \lambda^k$ with $k > 0$ then the action becomes:

$$\begin{aligned}&\Psi(\exp X_k)((\underline{i}_1\wedge\underline{i}_2\cdots)^0,\ldots,(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^j,\ldots,(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^n)\\ &\quad=(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^0,\ldots,(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^{k-1},(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^k\\ &\qquad+(X_k(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^0)^k,\ldots,\left(\sum_{q=0}^{[r/k]}\frac{1}{q!}X_k^q(\underline{i}_1\wedge\underline{i}_2\wedge\cdots)^{r-kq}\right)^r,\ldots,\\ &\qquad\times\left(\left(\sum_{q=0}^{[n/k]}\frac{1}{q!}X_k^q\underline{i}_1\wedge\underline{i}_2\wedge\cdots\right)^{n-kq}\right)^n\end{aligned}\tag{4.19}$$

where we have denoted with $[\frac{r}{k}]$ the integer part of $\frac{r}{k}$.

It is still possible to construct a bosonization of the representation Ψ of $\mathfrak{gl}_{\infty}^{(n)}$, which generalize to the our contest that already known in the literature [10,11,13]. To achieve this task we have first to extend the representation Ψ from $\mathfrak{gl}_{\infty}^{(n)}$ to $\mathfrak{a}_{\infty}^{(n)} = \mathfrak{a}_{\infty} \otimes \mathbb{C}^{(n)}$, this requires to get rid from anomalies to modify our representation Ψ by putting

$$\Psi(E_{ij}^k) = \begin{cases} \frac{1}{k+1} \sum_{l=0}^{k} \psi_i^{(k-l)} \psi_j^{*(l)} & \text{if } i \neq j \text{ or } i = j > 0 \\ -\frac{1}{k+1} \sum_{l=0}^{k} \psi_j^{*(l)} \psi_i^{(k-l)} & \text{if } i = j \leq 0. \end{cases} \tag{4.20}$$

Next we must define the subalgebra $\mathfrak{s}^{(n)}$ of $\mathfrak{a}_{\infty}^{(n)}$ spanned by the elements

$$s_i^k = \sum_{j \in \mathbb{Z}} E_{j,j+i}^k, \quad \text{and } c^k, \quad k = 0, \ldots, n$$

whose Lie brackets are

$$[s_p^k, s_q^j] = \begin{cases} p\delta_{p,-q} c^{k+j} & \text{if } j + k \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{4.21}$$

and which can be therefore viewed as a generalized “Heisenberg Lie algebra”. Using the representation Ψ this algebra is given by the free bosonic fields α_j^k:

$$\begin{aligned} &\alpha_j^k = \frac{1}{k+1} \sum_{l=0}^{k} \sum_{i \in \mathbb{Z}} \psi_i^{(k-l)} \psi_{i+j}^{*(l)} \qquad \text{if } j \in \mathbb{Z}\{0\}, \quad k = 0, \ldots, n \\ &\alpha_0^k = \frac{1}{k+1} \sum_{l=0}^{k} \sum_{i>0} \psi_i^{(k-l)} \psi_i^{*(l)} - \sum_{l=0}^{k} \sum_{i \leq 0} \psi_i^{(k-l)} \psi_i^{*(l)}, \quad k = 0, \ldots, n. \end{aligned} \tag{4.22}$$

Now following Kac in [10] we introduce the bosonic Fock space $B^{(n)}$ given by the direct of $n+1$ copies of the usual bosonic Fock space $B = \mathbb{C}[x_1, x_2, \ldots; q, q^{-1}]$:

$$B^{(n)} = \bigoplus_{i=0}^{n} B_i \tag{4.23}$$

where B_I $i = 0, \ldots, n$ are copies of B. For our purposes it is also useful to look at this space as the tensor product between the Fock space B and an $n+1$ dimensional complex space:

$$B^{(n)} = B \otimes \mathbb{C}^{(n+1)} \tag{4.24}$$

and to consider its decomposition in “charged subspaces”:

$$B^{(n)} = \bigoplus_{m \in \mathbb{Z}} B_m, \qquad B_m = q^m \mathbb{C}[x_1, x_2, \ldots] \otimes \mathbb{C}^m. \tag{4.25}$$

On it is defined the following representation $r^{B^{(n)}}$ of the generalized Heisenberg algebra $\mathfrak{s}^{(n)}$ where we have normalized c to 1:

$$\begin{aligned} &\left.\begin{aligned} r^{B^{(n)}}(s_m^k) &= \frac{\partial}{\partial x_m} \otimes \Lambda^k \\ r^{B^{(n)}}(s_{-m}^k) &= m x_m \otimes \Lambda^k \end{aligned}\right\}, \text{ if } m > 0, k = 0, \ldots, n \\ &r^{B^{(n)}}(s_0^k) = q \frac{\partial}{\partial q} \otimes \Lambda^k, \qquad\qquad k = 0, \ldots, n. \end{aligned} \tag{4.26}$$

It is straightforward to see that the usual isomorphism of $\mathfrak{s} = \mathfrak{s}^{(0)}$-modules $\sigma : F \simeq B$ (see [10]) can be extended to a $\mathfrak{s}^{(n)}$-modules $\sigma^n : F^{(n)} \simeq B^{(n)}$ simply by taking the direct sum of $n+1$ copies of the isomorphism σ.

To compute the action of the isomorphism σ is useful to introduce the generating series of the fermionic fields:

$$\psi^{(k)}(z)=\sum_{j\in\mathbb{Z}}\psi_j^{(k)}z^j,\qquad \psi^{*(k)}(z)=\sum_{j\in\mathbb{Z}}\psi_j^{*(k)}z^{-j},\quad k=0,\ldots,n \tag{4.27}$$

and also the corresponding bosonic operators:

$$\Gamma_+^k(z)=\sum_{n\geq 1}\alpha_n^k\frac{z^{-n}}{n},\qquad \Gamma_-^k(z)=\sum_{n\geq 1}\alpha_{-n}^k\frac{z^n}{n},\quad k=0,\ldots,n. \tag{4.28}$$

Which can be also written as:

$$\Gamma_+^k(z)=\sum_{n\geq 1}\frac{z^{-n}}{n}\frac{\partial}{\partial x_n}\otimes\Lambda^k,\qquad \Gamma_-^k(z)=\sum_{n\geq 1}\frac{z^n}{n}x_n\otimes\Lambda^k. \tag{4.29}$$

Using these operators we have indeed the

Theorem 4.4. *For every $k=0,\ldots,n$ we have*:

$$\begin{aligned}\psi^{(k)}(z)&=z^{\alpha_0^k}q\Gamma_-^k(z)\Gamma_+^k(z)^{-1}\\ \psi^{*(k)}(z)&=q^{-1}z^{-\alpha_0^k}\Gamma_-^k(z)^{-1}\Gamma_+^k(z).\end{aligned} \tag{4.30}$$

Proof. Let us prove only the first of equations (4.30) since a completely similar construction works for the second ones. From the Eqs. (4.22) and (4.27) we obtain that

$$[\alpha_j^k,\psi^{(i)}(z)]=\begin{cases}z^j\psi^{(k+i)}(z) & \text{if } i+k\leq n\\ 0 & \text{otherwise}\end{cases}$$

$$[\alpha_j^k,\psi^{*(i)}(z)]=\begin{cases}-z^j\psi^{*(k+i)}(z) & \text{if } i+k\leq n\\ 0 & \text{otherwise}\end{cases}$$

now using the map $\sigma^{(n)}$ we can transport these relations to $B^{(n)}$, for $j>0$ we have

$$\begin{aligned}\sigma^{(n)}[\alpha_j^k,\psi^{(i)}(z)](\sigma^{(n)})^{-1}&=\left[\frac{\partial}{\partial x_j}\otimes\Lambda^k,\sigma^{(n)}\psi^{(i)}(z)(\sigma^{(n)})^{-1}\right]\\ &=\begin{cases}z^j\sigma^{(n)}\psi^{(k+i)}(z)(\sigma^{(n)})^{-1} & \text{if } i+k\leq n\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

while for j negative

$$\begin{aligned}\sigma^{(n)}[\alpha_j^k,\psi^{(i)}(z)](\sigma^{(n)})^{-1}&=[x_j\otimes\Lambda^k,\sigma^{(n)}\psi^{(i)}(z)(\sigma^{(n)})^{-1}]\\ &=\begin{cases}\frac{z^{-j}}{j}\sigma^{(n)}\psi^{(k+i)}(z)(\sigma^{(n)})^{-1} & \text{if } i+k\leq n\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Using these equations, the fact that $\psi_j^{(k)}$ can be written as $\psi_j \Lambda^k$ and Lemma 14.5 of [10] we can now conclude that the operator $\sigma^{(n)}\psi_j^{(k)}(\sigma^{(n)})^{-1}$ brings the subspace $B_m^{(n)}$ in the subspace $B_{m+1}^{(n)}$ for every m and it is of the form

$$\sigma^{(n)}\psi_j^{(k)}(\sigma^{(n)})^{-1} = C_m(z) q \Gamma^k(z)$$

with

$$\Gamma^k(z) = \left\{ \exp\left(\sum_{j\geq 1} z^j x_j\right) \exp\left(-\sum_{j\geq 1} \frac{z^{-j}}{j}\frac{\partial}{\partial x_j}\right)\right\} \otimes \Lambda^k$$

while the same argument used in the proof of Theorem 14.10 in [10] shows that $C_m(z) = z^{m+1}$. $\square$

Theorem 4.5. *The generating series for the representation* Ψ (4.17) *of* $\mathfrak{gl}_\infty^{(n)}$ *is*

$$\sum_{i,j\in\mathbb{Z}} z_1^i z_2^{-j} \Psi(E_{ij}^k) = \left(\frac{z_1}{z_2}\right)^m \frac{1}{1-(z_2/z_1)} \Gamma^k(z_1, z_2) \tag{4.31}$$

where

$$\Gamma^k(z_1, z_2) = (k+1) \exp\left(\sum_{p\geq 1} (z_1^p - z_2^p) x_p\right) \exp\left(\sum_{p\geq 1} \frac{z_1^{-p} - z_2^{-p}}{p} \frac{\partial}{\partial x_p}\right) \Lambda^k. \tag{4.32}$$

Proof. We observe that from (4.27) follows

$$\sum_{i,j\in\mathbb{Z}} z_1^i z_2^{-j} \Psi(E_{ij}^k) = \frac{1}{k+1} \sum_{l=0}^{k} \psi_i^{(k-l)}(z_1) \psi_j^{*(l)}(z_2)$$

substituting (4.30) we get

$$\sum_{i,j\in\mathbb{Z}} z_1^i z_2^{-j} \Psi(E_{ij}^k) = \frac{1}{k+1} \sum_{l=0}^{k} z_1^{\alpha_0^{k-l}} q \Gamma_-^{k-l}(z_1) \Gamma_+^{k-l}(z_1)^{-1} q^{-1} z_2^{-\alpha_0^l} \Gamma_-^l(z_2)^{-1} \Gamma_+^l(z_2).$$

Since it holds [10] for every $0 \leq l \leq k$ that

$$\Gamma_+^{k-l}(z_1)^{-1} \Gamma_-^l(z_2)^{-1} = \Gamma_-^l(z_2)^{-1} \Gamma_+^{k-l}(z_1)^{-1} \left(1 - \frac{z_2}{z_1}\right)^{-1}$$

the previous equation using also (4.29) becomes

$$\left(1-\frac{z_2}{z_1}\right)^{-1}\frac{1}{k+1}\sum_{l=0}^{k}\sum_{s=0}^{n-k+l} z_1^m e_{k-l+s,s}\sum_{r=0}^{n-l} z_2^{-m} e_{l+s,s}$$
$$\times\sum_{s=0}^{n-k+l}\exp\left(\sum_{p\geq 1}(z_1^p x_p)\right) e_{k-l+s,s}\sum_{r=0}^{n-l}\exp\left(\sum_{p\geq 1}-z_2^p x_p\right) e_{l+r,r}$$
$$\times\sum_{s=0}^{n-k+l}\exp\left(\sum_{p\geq 1}\frac{z_1^{-p}(\partial/\partial x_p)}{p}\right) e_{k-l+s,s}\sum_{r=0}^{n-l}\exp\left(\sum_{p\geq 1}\frac{-z_1^{-p}(\partial/\partial x_p)}{p}\right) e_{l+r,r}$$

and finally

$$(k+1)\left(\frac{z_1}{z_2}\right)^m\left(1-\frac{z_2}{z_1}\right)^{-1}$$
$$\times\exp\left(\sum_{p\geq 1}(z_1^p - z_2^p)x_p\right)\exp\left(\sum_{p\geq 1}\frac{z_1^{-p}-z_2^{-p}}{p}\frac{\partial}{\partial x_p}\right)\Lambda^k \quad \square$$

5. Coupled Hirota bilinear equations

The aim of this section is to derive from the vertex operator algebras constructed in the previous one the corresponding hierarchies of Hirota bilinear equations. The key link to connect our representations with the corresponding bilinear equations are opportune homogeneous Casimir operators acting on particular quotient of tensor products of representations. The starting point is to observe that the representations of $\mathfrak{g}^{(n)}$ or $\mathfrak{gl}_\infty^{(n)}$ and $CL^{(n)}$ presented in the previous section can be viewed as tensor product between an infinite dimensional space V and the space $\mathbb{C}^{n+1}$ where $\tilde{\mathcal{L}}(\mathfrak{g})$ and d (or $\mathfrak{gl}_\infty$ and CL) and the polynomial ring $\mathbb{C}^{(n)}(\lambda)$ respectively act (as already pointed out in Eq. (4.1) for the algebras $\mathfrak{g}^{(n)}$). Moreover since it is easily to see that $\mathbb{C}^{n+1}$ as $\mathbb{C}^{(n)}(\lambda)$-module is isomorph to $\mathbb{C}^{(n)}(\lambda)$ itself, we can decompose our representation's space as the tensor product:

$$V^{(n)}(\lambda) = V\otimes\mathbb{C}^{(n)}(\lambda). \tag{5.1}$$

This in turn being $V\otimes\mathbb{C}^{(n)}(\lambda)$ a $\mathbb{C}^{(n)}(\lambda)$-module with trivial action on the first factor allow us to construct a $\mathfrak{g}^{(n)}$ ($\mathfrak{gl}_\infty$ and $CL^{(n)}$) representation on the "modified tensor product"

$$(V\otimes\mathbb{C}^{(n)}(\lambda))\otimes_{\mathbb{C}^{(n)}(\lambda)}(V\otimes\mathbb{C}^{(n)}(\lambda)), \tag{5.2}$$

these will be the space where our generalized Hirota equations will live. Since in a similar way also the algebras $\mathfrak{g}^{(n)}$, $\mathfrak{gl}_\infty^{(n)}$, and $CL^{(n)}$ are $\mathbb{C}^{(n)}(\lambda)$-modules we may consider the following elements of $\mathfrak{g}^{(n)}\otimes_{\mathbb{C}^{(n)}(\lambda)}\mathfrak{g}^{(n)}$, $\mathfrak{gl}_\infty^{(n)}\otimes_{\mathbb{C}^{(n)}}\mathfrak{gl}_\infty^{(n)}$ and $CL_\infty^{(n)}\otimes_{\mathbb{C}^{(n)}} CL^{(n)}$, respectively:

$$\begin{aligned}\Omega_2 &= \sum_{k,l=0}^{n} \sum_{\alpha\in\Delta\cup\{0\}} \sum_{i\in\mathbb{Z}} (e_{\alpha}^{(i)} \otimes \lambda^k) \otimes_{\mathbb{C}^{(n)}(\lambda)} (e_{-\alpha}^{(i)} \otimes \lambda^l) \\ &= \sum_{k=-0}^{n} \sum_{\alpha\in\Delta\cup\{0\}} \sum_{i\in\mathbb{Z}} (e_{\alpha}^{(i)} \otimes e_{-\alpha}^{(i)}) \otimes \lambda^k\end{aligned}$$

and

$$\Omega_1 = \sum_{k,l=0}^{n} \sum_{j\in\mathbb{Z}} (\psi_j \otimes \lambda^k) \otimes_{\mathbb{C}^{(n)}(\lambda)} (\psi_j^* \otimes \lambda^l) = \sum_{k=0}^{n} \sum_{j\in\mathbb{Z}} (\psi_j \otimes \psi_j^*) \otimes \lambda^k$$

which act naturally on the space $(V \otimes \mathbb{C}^{(n)}(\lambda)) \otimes_{\mathbb{C}^{(n)}(\lambda)} (V \otimes \mathbb{C}^{(n)}(\lambda))$.

Proposition 5.1. *The operators Ω_2 and Ω_1 commute with the action on $V \otimes \mathbb{C}^{(n)}(\lambda)$ of $\mathfrak{g}^{(n)}$ and $\mathfrak{gl}_{\infty}^{(n)}$, respectively.*

Proof. The thesis follows by straightforward computation. For example let us consider Ω_2 and let $X \otimes \lambda^j$ an homogeneous element of $\mathfrak{g}^{(n)}$ then we have indeed:

$$\begin{aligned}[\Omega_2, X \otimes \lambda^j] &= \left[\sum_{k=-0}^{n} \sum_{\alpha\in\Delta\cup\{0\}} \sum_{i\in\mathbb{Z}} (e_{\alpha}^{(i)} \otimes e_{-\alpha}^{(i)}) \otimes \lambda^k, X \otimes \lambda^j\right] \\ &= \left[\sum_{k=-0}^{n} \sum_{\alpha\in\Delta\cup\{0\}} \sum_{i\in\mathbb{Z}} (e_{\alpha}^{(i)} \otimes e_{-\alpha}^{(i)}), X\right] \otimes \lambda^{k+j} = 0\end{aligned}$$

because $\sum_{k=-0}^{n} \sum_{\alpha\in\Delta\cup\{0\}} \sum_{i\in\mathbb{Z}} e_{\alpha}^{(i)} \otimes e_{-\alpha}^{(i)}$ is Casimir operator of $\tilde{\mathcal{L}}(\mathfrak{g})$. Similar computations proves the statement for the operator Ω_1. □

5.1. Coupled KP hierarchies

Let us compute these equations explicitly starting with the case of $\mathfrak{gl}_{\infty}^{(n)}$.

Since $F^{(n)}(\lambda) \otimes_{\mathbb{C}^{(n)}(\lambda)} F^{(n)}(\lambda)$ is a $\mathfrak{gl}_{\infty}^{(n)}$-module the Casimir operator Ω_1 commutes with action of any element of this algebra and therefore with each element of $GL_{\infty}^{(n)}$. But this in turn observing that

$$\sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{j\in\mathbb{Z}} \frac{1}{k+1} (\psi_j \otimes \lambda^{k-l}) \otimes_{\mathbb{C}^{(n)}(\lambda)} (\psi_j^* \otimes \lambda^l) = \sum_{k=0}^{n} \sum_{j\in\mathbb{Z}} (\psi_j \otimes \psi_j^*) \otimes \lambda^k = \Omega_1$$

says that any vector $\tau = (\tau_0, \ldots, \tau_n)$ of the orbit of $GL_\infty^{(n)}(|0\rangle, 0, \ldots, 0)$ satisfies the equation

$$\sum_{\substack{p,q=0\\p+q\le n}}^{n} \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{j\in\mathbb{Z}} \frac{1}{k+1} \psi_j^{(k-l)}(\tau_p) \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi_j^{*(l)}(\tau_q) = 0. \tag{5.3}$$

Moreover the argument of Theorem 14.11 in [10] proves

Lemma 5.2. *The orbit of $GL_\infty^{(n)}|0\rangle$ is the set of all nonzero solutions $\tau \in F_0^{(n)}$ of Eq. (5.3).*

Our generalized Hirota bilinear equations will be the bosonic version of Eq. (5.3). For the convenience of the reader let us briefly outline their explicit construction despite to the fact that this is up to some minor changes quite standard. To apply to Eq. (5.3) the isomorphism $\sigma^{(n)}$ we have to write it in terms of $\psi^{(k)}(z)$ and $\psi^{*(k)}(z)$ as

$$z^0\text{-term of } \sum_{\substack{p,q=0\\p+q\le n}}^{n} \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{1}{k+1} \psi^{(k-l)}(z)\tau_p \otimes_{\mathbb{C}^{(n)}(\lambda)} \psi^{*(l)}(z)\tau_q = 0. \tag{5.4}$$

Then its bosonizated form is

$$\begin{aligned}&\operatorname{res}_{z=0} \sum_{\substack{p,q=0\\p+q\le n}}^{n} \sum_{k=0}^{n} \sum_{l=0}^{k} \left(\exp \sum_{j\ge 1} z^j (x'_j - x''_j)\right) \left(\exp - \sum_{j\ge 1} \frac{z^{-j}}{j} \left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j}\right)\right)\\ &\quad \times \tau_p(x')\tau_q(x'') = 0.\end{aligned} \tag{5.5}$$

Introducing the new variables

$$x_j = \tfrac{1}{2}(x'_j + x''_j), \qquad y_j = \tfrac{1}{2}(x'_j - x''_j)$$

Eq. (5.5) becomes:

$$\begin{aligned}&\operatorname{res}_{z=0} \sum_{\substack{p,q=0\\p+q\le n}}^{n} \sum_{k=0}^{n} \sum_{l=0}^{k} \left(\exp 2 \sum_{j\ge 1} z^j (y_j)\right) \left(\exp - \sum_{j\ge 1} \frac{z^{-j}}{j} \left(\frac{\partial}{\partial y_j}\right)\right)\\ &\quad \times \tau_p(x+y)\tau_q(x-y) = 0.\end{aligned}$$

This latter equation can be easily written in terms of elementary Schur polynomials $S_j\, j \in \mathbb{N}$ as:

$$\sum_{p=0}^{k} \sum_{j\ge 0} S_j(2y) S_{j+1}(-\tilde{\partial}_y) \tau_p(x+y)\tau_{k-p}(x-y) = 0, \qquad k = 0, \ldots, n \tag{5.6}$$

where as usual $\tilde{\partial}_y$ means $\left(\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, \ldots\right)$. Then introducing the Hirota bilinear differentiation by:

$$\begin{aligned}P(D_1, D_2, \ldots) fg = P\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots\right) f(x_1 + u_1, x_2 + u_2, \ldots)\\ \times g(x_1 - u_1, x_2 - u_2, \ldots)\end{aligned}$$

and using the Taylor formula

$$P(\tilde{\partial}_y)\tau_p(x+y)\tau_q(x-y) = P(\tilde{\partial}_u)\tau_p(x+y+u)\tau_q(x-y-u)|_{u=0}$$
$$= P(\tilde{\partial}_u)\left(\exp\sum_{j\geq 1} y_j\frac{\partial}{\partial u_j}\right)\tau_p(x+u)\tau_q(x-u)|_{u=0},$$

we can write (5.6) in the Hirota bilinear form:

$$\sum_{p=0}^{k}\sum_{j\geq 0} S_j(2y)S_{j+1}(-\tilde{D})\left(\exp\sum_{s\geq 1} y_s D_s\right)\tau_p\tau_{k-p}, \quad k=0,\ldots,n. \tag{5.7}$$

(Here again as usual $\tilde{D}$ stands for $(D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \ldots, 0)$). Expanding (5.7) as a multiple Taylor series in the variables $y_1, y_2, \ldots$ we obtain that each coefficient of the series must vanish giving arise to a hierarchy in an infinite number of non linear partial differential equations in the Hirota bilinear form, which of course contains the celebrated KP hierarchy. Observe that $P(D_1,\ldots,D_k)\sum_{p=0}^{k}\tau_p\tau_{k-p}=0$ identically for any odd monomial $P(D_1,\ldots,D_k)$ in the Hirota operators D_k because $\sum_{p=0}^{k}\tau_p\tau_{k-p}=\sum_{p=0}^{k}\tau_{k-p}\tau_p$ for any $k=0,\ldots,n$. Therefore the first non trivial coupled Hirota equations are:

$$\begin{aligned}
(D_1^4+3D_2^2-4D_1D_3)\tau_0\tau_0 &= 0\\
(D_1^4+3D_2^2-4D_1D_3)\tau_0\tau_1 &= 0\\
\cdots &= \cdots\\
(D_1^4+3D_2^2-4D_1D_3)\left(\textstyle\sum_{p=0}^{k}\tau_p\tau_{k-p}\right) &= 0\\
\cdots &= \cdots\\
(D_1^4+3D_2^2-4D_1D_3)\left(\textstyle\sum_{p=0}^{n}\tau_p\tau_{n-p}\right) &= 0
\end{aligned} \tag{5.8}$$

To write this equations in the "soliton variables" we perform the change of variables $u_0 = 2\frac{\partial^2\log(\tau_0)}{\partial x^2}$, $u_i=\frac{\tau_i}{\tau_0}$ which generalizes to our case those proposed by Hirota et al. in [8]. In these new variables equations (5.8) read

$$\begin{cases}
\frac{3}{4}u_{0yy}-(u_{0t}-\frac{3}{2}u_0u_{0x}-\frac{1}{4}u_{0xxx})_x=0\\
u_{kxxxx}-4u_{kxt}+3u_{kyy}+6u_0u_{kxx}+\Big(\sum_{j=1}^{k-1}2u_{jx}u_{(k-j)t}+2u_{jt}u_{(k-j)x}\\
\quad -3u_{jy}u_{(k-j)y}-6u_0u_{jx}u_{(k-j)x}-2u_{jxxx}u_{(k-j)x}-3u_{jxx}u_{(k-j)xx}\\
\quad -2u_{jx}u_{(k-j)xxx}\Big)=0, \quad k=1,\ldots,n
\end{cases} \tag{5.9}$$

where $x=x_1$, $y=x_2$ and $t=x_3$. The vertex operator construction offers a canonical way to produce a class of generalized soliton solutions for these equations. Indeed for what said above the group $G_\infty^{(n)}$ brings solutions into solutions. Therefore, in particular, we can construct non trivial solutions (called N soliton solution) of our hierarchies by acting with the polynomial vertex operators on the trivial solution $(1,0,\ldots,0)^{\mathrm{T}}$. This requires to write explicitly a formula for the composition of the action of N vertex operators on the space $B^{(n)}$. But using the induction on N and the identity $\exp(-\sum_{j\geq 1}x^j/j)=1-x$ this formula written in components, for a generic element $(\tau_0,\ldots,\tau_n)$ of $B^{(n)}$ and for some indeterminates $u_1^j,\ldots,u_N^j, v_1^j,\ldots,v_N^j$, $j=0,\ldots,n$ becomes

$$\left(\prod_{j=N}^{1}\left(\sum_{k_j=0}^{n}\Gamma(u_j^{k_j},v_j^{k_j})\Lambda^{k_j}\right)(\tau_0(x_1,x_2,\ldots),\ldots,\tau_n(x_1,x_2,\ldots))^{\mathrm{T}}\right)_m$$
$$=\sum_{\substack{k_1,\ldots,k_N,s=0\\k_1+\cdots+k_N+s=m}}^{n}\prod_{1\le i\le j\le N}\left[\frac{(u_j^{(k_j)}-u_i^{(k_i)})(v_j^{(k_j)}-v_i^{(k_i)})}{(u_j^{(k_j)}-v_i^{(k_i)})(v_j^{(k_j)}-u_i^{(k_i)})}\right.$$
$$\left.\times\left(\exp\sum_{r\ge1}\sum_{l=1}^{N}((u_l^{k_l})^r-(v_l^{k_l})^r)x_r\right)\times\tau_s\left(\ldots,x_r-\frac{1}{r}\sum_{l=1}^{N}((u_l^{k_l})^{-r}-(v_l^{k_l})^{-r}),\ldots\right)\right],$$
$$m=0,\ldots,n. \tag{5.10}$$

First of all this formula shows that any matrix $\sum_{k=0}^{n}\Gamma(u^k,v^k)\Lambda^k$ acts as nilpotent operator, it hold indeed

Lemma 5.3. *For every s, $0\le s\le n$ we have that*

$$\left(\sum_{k=0}^{n}\Gamma(u^k,v^k)\Lambda^k\right)^s(0,\ldots,\tau_r,0,\ldots)^{\mathrm{T}}=0 \tag{5.11}$$

for every $\tau_r\in\mathbb{C}(x_1,x_2,\ldots)$ and every choice of u^k and v^k, if and only if $s>\left[\frac{1+\sqrt{1+8(n-r)}}{2}\right]$ where $[x]$ denotes the integer part of x.

Moreover the mth component of the vector $(\sum_{k=0}^{n}\Gamma(u^k,v^k)\Lambda^k)^s(0,\ldots,\tau_r,0,\ldots)^{\mathrm{T}}$ vanishes identically if and only if $s>\left[\frac{1+\sqrt{1+8(m-r)}}{2}\right]$.

Proof. If we set in formula (5.10) $u_j^{k_j}=u^{k_j}$ and $v_j^{k_j}=v^{k_j}$ then it is easily to check that $(\sum_{k=0}^{n}\Gamma(u^k,v^k)\Lambda^k)^s(0,\ldots,\tau_r,0,\ldots)^{\mathrm{T}}=0$ for every $\tau_s\in\mathbb{C}(x_1,x_2,\ldots)$ and every choice of u^{k_j} and v^{k_j} if and only if $k_j=k_i$ for some k_j and k_i in each set of non positive integers $\{k_1,\ldots,k_s\}$ which appears in the right hand of (5.10). Or in other words if and only if any set of non negative integers $\{k_1,\ldots,k_s\}$ such that $\sum_{i=1}^{s}k_i=n-r$ contains at least two elements which coincide. Suppose that $\{k_1,\ldots,k_s\}$ is a sequence with all elements distinct such that $\sum_{i=1}^{s}k_i=n-r$, then, since the sequence of s non negative pairwise distinct integers whose sum is the smallest is obviously $\{0,1,\ldots,N\}$, we must have that $n-r=\sum_{i=1}^{s}k_i\ge\sum_{i=0}^{s}i=\frac{s(s-1)}{2}$. Therefore Eq. (5.11) is identically satisfied only and only if $s>\left[\frac{1+\sqrt{1+8(n-r)}}{2}\right]$. A completely similar argument proves the second part of the lemma. $\square$

Using the statement of the lemma we can write the exponential map of an element of the type $\sum_{k=0} \alpha_k \Gamma(u^k, v^k)$ as

$$\exp\left(\sum_{k=0} \alpha_k \Gamma(u^k, v^k)\right) = \sum_{k=0}^{n} \Lambda^k \left(\sum_{j=0}^{[(1+\sqrt{1+8k})/2]} \frac{1}{j!} \sum_{s_1+\cdots+s_j=k} \prod_{i=1}^{j} \alpha_{s_i} \Gamma(u_{s_i}, v_{s_i})\right).$$

Therefore from the Lemma 5.2 follows that the components of an N soliton solution of the polynomial KP hierarchy is

$$\begin{aligned} &\tau_{\alpha_0^1,\ldots,\alpha_n^1,\ldots,\alpha_n^N,u_0^1,\ldots,u_n^N,v_0^1,\ldots,v_n^N}(x) \\ &\quad = \sum_{k=0}^{n} \Lambda^k \left(\sum_{j=0}^{[(1+\sqrt{1+8k})/2]} \frac{1}{j!} \sum_{s_1+\cdots+s_j=k} \prod_{i=1}^{j} \alpha_{s_i} \Gamma(u_{s_i}, v_{s_i})\right)(1, 0, \ldots, 0)^{\mathrm{T}}. \end{aligned} \tag{5.12}$$

In particular an 1-soliton solution (again written in component) is

$$\begin{aligned} &(\tau_{\alpha_0,\ldots,\alpha_n,u_0,\ldots,u_n,v_0,\ldots,v_n}(x))_m \\ &\quad = \sum_{j=0}^{[(1+\sqrt{1+8k})/2]} \frac{1}{j!} \sum_{s_1+\cdots+s_j=k} \prod_{i=1}^{j} \alpha_{s_i} \prod_{0\le i<l\le j} \frac{(u_{s_i}-u_{i_l})(v_{s_i}-v_{s_l})}{(u_{s_i}-v_{s_l})(v_{s_i}-u_{s_l})} \\ &\qquad \times \left(\exp \sum_{r\ge 1} \sum_{i=1}^{j} (u_{s_i}^r - v_{s_i}^r) x_r\right), \qquad m = 0, \ldots, n. \end{aligned}$$

This solution for the simplest coupled case (when $n = 1$):

$$\begin{aligned} &\tfrac{3}{4} u_{0yy} - \left(u_{0t} - \tfrac{3}{2} u_0 u_{0x} - \tfrac{1}{4} u_{0xxx}\right)_x = 0 \\ &u_{1xxxx} - 4u_{1xt} + 3u_{1yy} + 6u_0 u_{1xx} = 0. \end{aligned} \tag{5.13}$$

takes with $\alpha_0 = \alpha_1 = 1$ the form

$$\begin{pmatrix} \tau_0 \\ \tau_1 \end{pmatrix} = \begin{pmatrix} 1 + \exp\left(\sum_{r\ge1}(u_0^r - v_0^r)x_r\right) \\ \exp\left(\sum_{r\ge1}(u_1^r - v_1^r)x_r\right) + 2\frac{(u_0-u_1)(v_0-v_1)}{(u_0-v_1)(v_0-u_1)} \\ \exp\left(\sum_{r\ge1}(u_0^r - v_0^r + u_1^r - v_0^r)x_r\right) \end{pmatrix}.$$

Of course in the contest of the single equations (5.13) we can view the indeterminates $x_4, x_5, \ldots$ as parameters in the expression of the solution, which will explicitly depend only from the first three ones. Therefore explicitly:

$$\begin{aligned}
u_0(x,y,t) &= \tfrac{1}{2}(u_0-v_0)\left(\cosh(\tfrac{1}{2}(u_0-v_0)x+(u_0^2-v_0^2)y+(u_0^3-v_0^3)t+\gamma_0)\right)^{-2}\\
u_1(x,y,t) &= \tfrac{1}{2}\left(\cosh(\tfrac{1}{2}(u_0-v_0)x+(u_0^2-v_0^2)y+(u_0^3-v_0^3)t+\gamma_0)\right)^{-1}\\
&\quad\times\left\{\mathrm{e}^{-1/2((u_0-v_0)x+(u_0^2-v_0^2)y+(u_0^3-v_0^3)t+\gamma_0)}\right.\\
&\quad\left.+2\tfrac{(u_0-u_1)(v_0-v_1)}{(u_0-v_1)(v_0-u_1)}\mathrm{e}^{1/2((u_0-v_0)x+(u_0^2-v_0^2)y+(u_0^3-v_0^3)t+\gamma_0)}\right\}\\
&\quad\times\mathrm{e}^{1/2((u_1-v_1)x+(u_1^2-v_1^2)y+(u_1^3-v_1^3)t+\gamma_1)}
\end{aligned}\tag{5.14}$$

where γ_i with $i=0,1$ are arbitrary constants.

5.2. Coupled KdV and Boussinesq hierarchies

Similarly we may construct a generalization of the KdV hierarchy (i.e., coupled KdV hierarchies) by considering the principal "basic" representation of the polynomial Lie algebra $\hat{\mathcal{L}}(\mathfrak{sl}_2^{(n)})$. From what done in Section 4 we consider the $\hat{\mathcal{L}}(\mathfrak{sl}_2^{(n)})$-module $V_Q^n = \oplus_{j=0}^n \mathbb{C}(x_1,x_3,x_5,\ldots)$ given by the formulas

$$\begin{aligned}
&H_j^k=\Lambda^k\tfrac{\partial}{\partial x_j},\quad H_{-j}^k=jx_j\Lambda^k, && j\in\mathbb{N}^{\mathrm{odd}},\ k=0,1,\ldots,n\\
&c_k=\Lambda^k,\quad 2d-\tfrac{1}{2}A_0^k=-\textstyle\sum_{j\in\mathbb{N}^{\mathrm{odd}}} jx_j\Lambda^k\tfrac{\partial}{\partial x_j}, && k=0,1,\ldots,n\\
&A^k(z)=\tfrac{1}{2}(\Gamma^k(z)-1), && k=0,1,\ldots,n
\end{aligned}\tag{5.15}$$

where

$$H_{2j+1}^k=t^j(X_\alpha^k-tX_{-\alpha}^k),\quad A_{2j}^k=-t^j(tH_\alpha^k),\quad A_{2j+1}^k=t^j(X_\alpha^k-tX_{-\alpha}^k)$$

with

$$X_\alpha^k=\begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes\lambda^k,\qquad X_{-\alpha}^k=\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\lambda^k,\qquad H_\alpha^k=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\lambda^k$$

and finally

$$\Gamma^k(z)=\left(\exp 2\sum_{j\in\mathbb{N}^{\mathrm{odd}}} z^jx_j\right)\left(\exp -2\sum_{j\in\mathbb{N}^{\mathrm{odd}}}\frac{z^{-j}}{j}\frac{\partial}{\partial x_j}\right)\Lambda^k.$$

Then the polynomial Hirota bilinear equation are given by

$$\Omega_2(v\otimes_{\mathbb{C}^{(n)}(\lambda)}v)=\mu v\otimes_{\mathbb{C}^{(n)}(\lambda)}v,$$

where $\mu \in \mathbb{C}$ which is equivalent to the following hierarchy of bilinear equations:

$$\sum_{p=0}^{k}\left(\sum_{j>0} S_j(4y_1, 0, 4y_3, \dots)S_j\left(-\frac{2}{1}D_1, 0, -\frac{2}{3}D_3, \dots\right) - 8\sum_{j\in\mathbb{N}^{\mathrm{odd}}} jy_jD_j\right)$$
$$\times\left(\exp\sum_{j\in\mathbb{N}^{\mathrm{odd}}} y_jD_j\right)\tau_p\tau_{k-p} = 0, \quad k = 0, \dots, n. \tag{5.16}$$

Reasoning as in the previous case of the coupled KP equations we have that the first non trivial bilinear equations in the hierarchy are:

$$\sum_{p=0}^{k}(-4D_1D_3 + D_1^4)\tau_p\tau_{k-p} = 0, \quad k = 0, \dots, n.$$

These equations, by imposing the variables' transformation $u_0 = 2\frac{\partial^2\log(\tau_0)}{\partial x^2}$ $u_i = \frac{\tau_i}{\tau_0}$ $i = 1, \dots, n$, become

$$\begin{cases} (u_{0t} - \frac{3}{2}u_0u_{0x} - \frac{1}{4}u_{0xxx})_x = 0 \\ u_{kxxxx} - 4u_{kxt} + 6u_0u_{kxx} + \Big(\sum_{j=1}^{k-1} 2u_{jx}u_{(k-j)t} + 2u_{jt}u_{(k-j)x} \\ \quad - 6u_0u_{jx}u_{(k-j)x} - 2u_{jxxx}u_{(k-j)x} - 3u_{jxx}u_{(k-j)xx} - 2u_{jx}u_{(k-j)xxx}\Big) = 0, \\ \quad k = 1, \dots, n. \end{cases} \tag{5.17}$$

which are generalizations of the coupled equation (12) in [8]. In particular for $n = 1$ we have

$$\begin{array}{l} u_{0t} - \frac{3}{2}u_0u_{0x} - \frac{1}{4}u_{0xxx} = 0 \\ 6u_0u_{1xx} + u_{1xxxx} - 4u_{1xt} = 0. \end{array} \tag{5.18}$$

This latter equations make the contact with the literature [8] and [20] (more precisely setting $u = u_0$, $v = u_{1x}$ and rescaling the time $t \to -4t$ one obtains equations (2) of [20]). Further is worth to note that by taking the derivative with respect to x of the second equation and putting $v_0 = u_0$ and $v_1 = u_{1xx}$ equations (5.18) become

$$v_{0t} = \frac{3}{2}v_0v_{0x} + \frac{1}{4}v_{0xxx}$$
$$v_{1t} = \frac{1}{4}v_{1xxx} + \frac{3}{2}v_0v_{1x} + \frac{3}{2}v_{0x}v_1.$$

These equations are bihamiltonian with respect the two Poisson tensors [1,5]

$$P_1 = \begin{pmatrix} \frac{1}{2}\partial_{xxx} + 2v_0\partial_x + v_{0x} & 0 \\ 0 & -2\partial_x \end{pmatrix}$$
$$P_2 = \begin{pmatrix} 0 & \frac{1}{2}\partial_{xxx} + 2v_0\partial_x + v_{0x} \\ \frac{1}{2}\partial_{xxx} + 2v_0\partial_x + v_{0x} & 2v_1\partial_x + v_{1x} \end{pmatrix}$$

namely

$$\begin{pmatrix} v_{0t} \\ v_{1t} \end{pmatrix} = P_1 \begin{pmatrix} -\frac{1}{2}v_0 \\ -\frac{1}{8}v_{1xx} - \frac{3}{4}v_0 v_1 \end{pmatrix} = P_2 \begin{pmatrix} \frac{1}{2}v_1 \\ -\frac{1}{2}v_0 \end{pmatrix}.$$

Similarly they can be also written in the Lax form $\frac{\mathrm{d}L}{\mathrm{d}t} = [L, B]$ where

$$L = \begin{pmatrix} \partial_{xx} + v_0 & 0 \\ v_1 & \partial_{xx} + v_0 \end{pmatrix}$$
$$B = \begin{pmatrix} -\partial_{xxx} - \frac{3}{4}v_{0x} - \frac{3}{2}v_0\partial_x & 0 \\ -\frac{3}{4}v_{1x} - \frac{3}{2}v_1\partial_x & -\partial_{xxx} - \frac{3}{4}v_{0x} - \frac{3}{2}v_0\partial_x \end{pmatrix}.$$

Moreover analogous changes of variables lead to the Lax pairs for the other hierarchies arising from the Lie algebras $\hat{\mathcal{L}}(\mathfrak{sl}_k^{(n)})$.

Actually, as in the standard case, these hierarchy can be recovered from the polynomial KP (5.8) by performing a reduction procedure, which amounts to eliminate the dependence from the "even" variables x_{2j} $j \in \mathbb{N}$ of the Fock space. This in turn corresponds to restrict the representation of $\mathfrak{gl}_\infty^{(n)}$ onto its subalgebras $\hat{\mathcal{L}}(\mathfrak{sl}_2^{(n)})$, giving Lie algebraic explanation of what done in the recent literature [14]. Therefore the soliton solutions for the coupled KdV hierarchies can be recovered from those written for the coupled KP equations (5.12) erasing the even variables. In the particular case whereas $n = 2$ this reduction method applied to (5.14) leads to the following solutions:

$$\begin{aligned} u_0(x, y, t) &= \tfrac{1}{2}(u_0 - v_0)\left(\cosh(\tfrac{1}{2}(u_0 - v_0)x + (u_0^3 - v_0^3)t + \gamma_0)\right)^{-2} \\ u_1(x, y, t) &= \tfrac{1}{2}\left(\cosh\left(\tfrac{1}{2}(u_0 - v_0)x + (u_0^3 - v_0^3)t + \gamma_0\right)\right)^{-1} \\ &\quad \times \left\{ \mathrm{e}^{-1/2((u_0-v_0)x+(u_0^3-v_0^3)t+\gamma_0)} \right. \\ &\quad \left. + 2\frac{(u_0-u_1)(v_0-v_1)}{(u_0-v_1)(v_0-u_1)}\mathrm{e}^{1/2((u_0-v_0)x+(u_0^3-v_0^3)t+\gamma_0)} \right\} \\ &\quad \times \mathrm{e}^{1/2((u_1-v_1)x+(u_1^3-v_1^3)t+\gamma_1)} \end{aligned} \tag{5.19}$$

where γ_i with $i = 0, 1$ are still arbitrary constants.

In the same way we can recovered from the coupled KP hierarchy the "coupled Boussinesq" hierarchy by erasing all the variables x_{3j} with $j \in \mathbb{N}$, which again corresponds to restrict our representation to the Lie algebra $\hat{\mathcal{L}}(\mathfrak{sl}_3^{(n)})$. In this case the first non trivial bilinear Hirota equations are:

$$(D_1^4 + 3D_2^2)\left(\sum_{p=0}^{n} \tau_p \tau_{n-p}\right) = 0, \quad k = 0, \ldots, n.$$

In particular when $n = 1$ putting $u_0 = 2(\log(\tau_0))_{xx}$ and as usual $u_1 = \frac{\tau_1}{\tau_0}$ we get

$$\begin{aligned}&3u_{0tt} + u_{0xxxx} + 6u_{0x}^2 + 6u_0u_{0xx} = 0\\ &3u_{1tt} + u_{1xxxx} + 6u_0u_{1xx} = 0\end{aligned}$$

where $x = x_1, t = x_2$. The multi-soliton solutions of these equation can obviously recovered from the solutions (5.12) by erasing the variables x_{3j}.

5.3. Coupled BKP hierarchies and their first reductions

The construction presented above can be extended to simple Lie algebras, which are not of type A. In particular we would like to finish the chapter by outlining briefly the case of the Lie algebras of type B. In order to construct the bilinear Hirota equations for the polynomial BKP hierarchy, we have to consider the polynomial Clifford algebra $\mathrm{CL}_B^{(n)}$ defined as $\mathrm{CL}_B \otimes \mathbb{C}^{(n)}(\lambda)$. As in the A case it can be seen as an algebra of operators on a particular space. Let V be the irreducible Verma module with highest weight vector $|0\rangle$ for the usual Clifford Lie algebra CL_B (i.e. $\mathrm{CL}_B^{(0)}$). Let us consider on $V^{(n)} = \oplus_{i=0}^n V_i$ $V_i \simeq V, \forall i = 0, \dots, n$ the operators $\phi_i^{(k)}$, $i \in \mathbb{Z}$, $k = 0, \dots, n$:

$$\phi_i^{(j)}(v_0, \dots, v_n) = (\underbrace{0, \dots, 0}_{j}, \phi_i v_0, \dots, \phi_i v_{n-j})$$

where $\phi_i v_j$ is the usual action of the elements of CL_B on V. From the tensorial definition of $\mathrm{CL}_B^{(n)}$ the relation among the elements of the algebra becomes in this polynomial case:

$$\phi_i^{(k)}\phi_l^{(j)} + \phi_l^{(j)}\phi_i^{(k)} = \begin{cases} (-1)^i \delta_{i,-l} \Lambda^{j+k} & \text{if } j + k \le n \\ 0 & \text{otherwise.} \end{cases}$$

Using this action we can define for $n \in \mathbb{Z}^{\text{odd}}$ the neutral bosonic fields:

$$\beta_m^k = \frac{1}{2(k+1)} \sum_{l=0}^{k} \sum_{j \ge 1} (-1)^{j+1} \phi_j^{(k-l)} \phi_{-j-m}^{(l)}$$

which generate the associated generalized Heisenberg algebra

$$[\beta_p^h, \beta_q^k] = \begin{cases} \frac{1}{2} p \delta_{p,-q} \Lambda^{h+k} & \text{if } h + k \le n \\ 0 & \text{otherwise.} \end{cases}$$

Now we can define a generalized boson–fermion correspondence of type B $\sigma_B^{(n)} : V^{(n)} \to B^{(n)} = \oplus_{k=0}^n B_k$ where $B_k = \mathbb{C}[x_1, x_3, x_5, \dots; q]/(q^2 - \frac{1}{2})$ for all k, which nothing else that the direct sum of $n+1$ copies of the usual isomorphism [10] σ_B and therefore

$$\sigma_B^{(n)}(\underbrace{0,\ldots,0}_{k-1},|0\rangle,0\ldots,0)=(\underbrace{0,\ldots,0}_{k-1},1,0\ldots,0)$$
$$\sigma_B^{(n)}\phi_0^k(|0\rangle,0\ldots,0)=(\underbrace{0,\ldots,0}_{k-1},q,0\ldots,0)$$

and for $p\in\mathbb{N}^{\mathrm{odd}}$

$$\sigma_B^{(n)}\beta_p^k(\sigma_B^{(n)})^{-1}=\Lambda^k\frac{\partial}{\partial x_p},\qquad \sigma_B^{(n)}\beta_{-p}^k(\sigma_B^{(n)})^{-1}=\frac{1}{2}\Lambda^k p x_p.$$

Then if we introduce the neutral fermionic fields:

$$\phi^{(k)}(z)=\sum_{i\in\mathbb{Z}}\phi_i^{(k)}z^i$$

we can show (as in the case of $\mathfrak{a}_\infty^{(n)}$) that

$$\sigma_B^{(n)}\phi^{(k)}(z)(\sigma_B^{(n)})^{-1}=\Lambda^k q\exp\left(\sum_{j\in\mathbb{N}^{\mathrm{odd}}}x_jz^j\right)\exp\left(-2\sum_{j\in\mathbb{N}^{\mathrm{odd}}}\frac{z^{-j}}{j}\frac{\partial}{\partial x_j}\right).$$

Our aim is now to construct a fermionic representation of the infinite dimensional polynomial Lie algebra $\mathfrak{so}_\infty^{(n)}=\mathfrak{so}_\infty\otimes\mathbb{C}^{(n)}(\lambda)$ (and actually of $\mathfrak{b}_\infty^{(n)}=\mathfrak{b}_\infty\otimes\mathbb{C}^{(n)}(\lambda)$) spanned by the elements $F_{ij}^k=(-1)^jE_{ij}^k-(-1)^iE_{-j,-i}^k$, where the E_{ij}^k are the basis of $\mathfrak{gl}_\infty^{(n)}$ previously considered. Mimicking the same proof of Theorem 4.2 one can prove indeed that the following formula:

$$\rho(F_{ij}^k)=\frac{1}{k+1}\sum_{l=0}^{k}\phi_i^{(k-l)}\phi_{-j}^{(l)}$$

defines a representation of $\mathfrak{so}_\infty^{(n)}$, which can be linearly extended to a representation of $\mathfrak{b}_\infty^{(n)}$ by putting

$$\hat\rho(F_{ij}^k)=\begin{cases}\frac{1}{k+1}\sum_{l=0}^{k}\phi_i^{(k-l)}\phi_{-j}^{(l)} & \text{if } i\neq j \text{ or } i=j>0\\ \frac{1}{k+1}\sum_{l=0}^{k}\phi_i^{(k-l)}\phi_{-j}^{(l)}-\frac{1}{2}\Lambda^k & \text{if } i=j<0\end{cases}$$
$$\hat\rho(c_k)=\Lambda^k,\qquad k=0,\ldots,n.$$

This representation turns out to be the direct sum of two representation defined respectively on $V_0^{(n)}$ (the even elements $V^{(n)}$) and on $V_1^{(n)}$ (the odd ones). Moreover it can be checked that the map $\sigma_B^{(n)}:V_0^{(n)}\simeq\oplus_{k=0}^{n}B_{k0}$ (where $B_{k0}=\mathbb{C}(x_1,x_2,x_3,\ldots)$ for all k) is a $\mathfrak{so}_\infty^{(n)}$-isomorphism between the representation $\rho_{|V_0}$ and the following vertex operator construction of the same algebra such that:

$$\sum_{i,j\in\mathbb{Z}} z_1^i z_2^{-j} F_{ij}^k \mapsto \frac{1}{2}\frac{1-z_2/z_2}{1+z_2/z_1}(\Gamma_B^k(z_1,z_2)-1)$$

where

$$\Gamma_B^k(z_1,z_2)=(k+1)\Lambda^k \exp\left(\sum_{j\in\mathbb{N}^{\mathrm{odd}}} x_j(z_1^j+z_2^j)\right)\exp\left(-2\sum_{j\in\mathbb{N}^{\mathrm{odd}}}\frac{z_1^{-j}-z_2^{-j}}{j}\frac{\partial}{\partial x_j}\right).$$

In order to construct the polynomial nBKP hierarchy of Hirota bilinear equation we use the operator

$$\Omega_1^B=\sum_{k=0}^n\frac{1}{k+1}\sum_{l=0}^k\sum_{j\in\mathbb{Z}}(-1)^j\phi_j^{(k-l)}\otimes_{\mathbb{C}^{(n)}(\lambda)}\phi_{-j}^{(l)}$$

commuting with the action of the algebra $\mathfrak{b}_\infty^{(n)}$. The equation on $V^{(n)}\otimes_{\mathbb{C}^{(n)}(\lambda)}V^{(n)}$

$$\Omega_1^B(\tau\otimes_{\mathbb{C}^{(n)}(\lambda)}\tau)=\sum_{k=0}^n\frac{1}{k+1}\sum_{l=0}^k(-1)^j\phi_0^{(k-l)}(\tau)\otimes_{\mathbb{C}^{(n)}(\lambda)}\phi_0^{(l)}(\tau),\qquad \tau\in V_0$$

transferred to $\oplus_{k=0}^n B_{k0}$ gives rise to the coupled BKP hierarchy

$$\sum_{p=0}^k\sum_{j\in\mathbb{N}^{\mathrm{odd}}} S_j(2y_j)S_j\left(-\frac{2}{j}D_j\right)\left(\exp\sum_{s\in\mathbb{N}^{\mathrm{odd}}} y_sD_s\right)\tau_p\tau_{k-p},\qquad k=0,\ldots,n. \tag{5.20}$$

For example the first non trivial ones (which therefore can be viewed as generalization to the B case of those written by Hirota) are the coefficients of y_6 in the expansion of (5.20):

$$\sum_{p=0}^k(D_1^6-5D_1D_3-5D_3^2+D_1D_5)\tau_p\tau_{k-p}=0,\qquad k=0,\ldots,n.$$

Performing the change of variables $w_0=2\frac{\partial\log(\tau_0)}{\partial x_1}$ and $w_1=\frac{\tau_i}{\tau_0}$ $i=1,\ldots,n$ these equations become

$$\begin{aligned}&(w_{0xxxxx}+30w_{0x}w_{0xxx}-5w_{0xxy}-30w_{0x}w_{0y}+60w_{0x}^3+9w_{0t})_x-5w_{0yy}=0\\&-5w_{1yy}+180w_{0x}w_{1xx}+9w_{1xt}+30w_{0xxx}w_{1xx}+30w_{0x}w_{1xxxx}+w_{1xxxxxx}\\&\quad-30w_{0x}w_{1xy}-30w_{0y}w_{1xx}-5w_{1xxxy}=0\end{aligned}$$

where $x=x_1$, $y=x_3$, $t=x_5$. Once again from these equations by performing opportune reduction process (namely eliminating the variable $x_{(2m+1)j}$) we can obtain the coupled

B_m soliton equations. In particular for $m = 1$ we get the coupled Kotera–Sawada hierarchy [22], whose first non trivial equation when $n = 1$ $w_0 = 2(\log(\tau_0))_{xx}$ and $w_1 = \frac{\tau_1}{\tau_0}$ is

$$\begin{aligned} &9w_{0t} + w_{0xxxxx} + 3w_{0x}w_{0xxx} + 3w_0 w_{0xxx} + 180w_0^2 w_{0x} = 0 \\ &180w_0 w_{1xx} + 9w_{1xt} + 30w_{0xx}w_{1xx} + 30w_0 w_{1xxxx} + w_{1xxxxxx} = 0 \end{aligned}$$

where $x = x_1$ and $t = x_5$.

Of course exactly as in the non coupled case these hierarchies can be also obtained by applying our construction to the Lie "polynomial" algebras $(A_1^{(2)})^{(n)}$. While for $m = 2$ one obtains the coupled B_2 hierarchies, which again when $n = 1$ has as first non trivial equation:

$$\begin{aligned} &(w_{0xxxxx} + 30w_{0x}w_{0xxx} - 5w_{0xxt} - 30w_{0x}w_{0t} + 60w_{0x}^3)_x - 5w_{0tt} = 0 \\ &-5w_{1tt} + 180w_{0x}w_{1xx} + 30w_{0xxx}w_{1xx} + 30w_{0x}w_{1xxxx} + w_{1xxxxxx} \\ &\quad - 30w_{0x}w_{1xt} - 30w_{0t}w_{1xx} - 5w_{1xxxt} = 0 \end{aligned}$$

where $x = x_1$ and $t = x_3$. Finally, the vertex operator construction provides (as in the case of the hierarchies of type A) the multi-soliton solutions for all the hierarchies written above.

Acknowledgements

The authors wish to thank Franco Magri for many helpful discussions and for the interest showed for our work. P.C. would like to express his warmful thanks to the participants of the Conference on Infinite Dimensional Lie Theory and its Applications (July 21–25, 2003) held in The Fields Institute of Toronto for very interesting discussions which have stimulated the first ideas for the present work and the members of Mathematics Department of the Chalmers University in Göteborg where part of the work has been done for their welcomed hospitality. G.O. thanks Sergio Cacciatori for useful discussions on the physical relevance of the integrable hierarchies.

References

[1] M. Antonowicz, A.P. Fordy, Factorisation of energy dependent Schrödinger operators: Miura maps and modified systems, Comm. Math. Phys. 124 (3) (1989) 465–486.

[2] Y. Billig, Sine-Gordon equation and representations of affine Kac–Moody algebra $\hat{\mathfrak{sl}}_2$, J. Funct. Anal. 192 (2) (2002) 295–318.

[3] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations, in: M. Jimbo, T. Miwa (Eds.), Proceedings of R.I.M.S. Symposium on Nonlinear Integrable Systems—Classical Theory and Quantum Theory, World Scientific, Singapore, 1983, pp. 39–119.

[4] V.G. Drinfeld, V.V. Sokolov, Lie algebras and equations of Korteweg–de Vries type, J. Sov. Math. 30 (1985) 1975–2036.

[5] A.P. Fordy, A.G. Reyman, M.A. Semenov-Tian-Shansky, Classical r-matrices and compatible Poisson brackets for coupled KdV systems, Lett. Math. Phys. 17 (1) (1989) 25–29.

[6] I. Frenkel, J. Lepowsky, A. Meurman, Vertex operator algebras and the Monster, Pure and Applied Mathematics, vol. 134, Academic Press, Inc., Boston, MA, 1988, liv + 508 pp., ISBN 0-12-267065-5.

[7] R. Hirota, Bilinear forms of soliton equations, Nonlinear Integrable Systems—Classical Theory and Quantum Theory (Kyoto, 1981), World Sci. Publishing, Singapore, 1983, pp. 15–37.

[8] R. Hirota, X. Hu, X. Tang, A vector potential KdV equation and vector Ito equation: soliton solutions, bilinear Bäcklund transformations and Lax pairs, J. Math. Anal. Appl. 288 (1) (2003) 326–348.
[9] J. Humprhreys, Introduction to Lie Algebras and Representation Theory (third printing), Revisited Springer Verlag, Heidelberg, 1980.
[10] V.G. Kac, Infinite dimensional Lie algebras, third ed., Cambridge University press, Cambridge, 1990.
[11] V.G. Kac, Vertex Algebras for Beginners, vol. 10, second ed., University Lecture Series, American Mathematical Society, Providence, RI, 1998, vi + 201 pp., ISBN 0-8218-1396-X.
[12] V.G. Kac, D.H. Peterson, Lectures on the infinite wedge-representation and the MKP hierarchy. Systémes dynamiques non linéaires: intégrabilité et comportement qualitatif, Sém. Math. Sup., vol. 102, Presses Univ. Montréal, Montreal, QC, 1986, pp. 141–184.
[13] V.G. Kac, M. Wakimoto, Exceptional hierarchies of soliton equations Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), Proc. Sympos. Pure Math., vol. 49, Am. Math. Soc., Providence, RI, 1989, pp. 191–237.
[14] S. Kakei, Dressing method and the coupled KP hierarchy, Phys. Lett. A 264 (6) (2000) 449–458.
[15] G. Magnano, F. Magri, Poisson-Nijenhuis structures and Sato hierarchy, Rev. Math. Phys. 3 (4) (1991) 403–466.
[16] A. Medina, P. Revoy, Algèbres de Lie et produit scalaire invariant, Ann. Sci. École Norm. Sup. (4) 18 (3) (1985) 553–561.
[17] A. Ooms, The Frobenius semiradical of a Lie algebra, J. Algebra 273 (1) (2004) 274–287.
[18] A. Pressley, G. Segal, Loop Groups, Oxford University, Oxford, 1986.
[19] M. Pedroni, P. Vanhaecke, A Lie algebraic generalization of the Mumford system, its symmetries and its multi-Hamiltonian structure, Regul. Chaotic Dyn. 3 (3) (1998) 132–160 (J. Moser at 70 (Russian)).
[20] S.Yu. Sakovich, Coupled KdV equations of Hirota-Satsuma type, J. Nonlinear Math. Phys. 6 (3) (1999) 255–262.
[21] M. Sato, The KP hierarchy and infinite-dimensional Grassmann manifolds, Theta functions–Bowdoin 1987, Part1 (Brunswick, ME, 1987), Proc. Sympos. Pure Math., vol. 49, Am. Math. Soc., Providence, RI, 1989, pp. 51–66.
[22] K. Sawada, T. Kotera, A method for finding *N*-soliton solutions of the K.d.V. equation and K.d.V.-like equation, Progr. Theor. Phys. 51 (1974) 1355–1367.
[23] M. Sato, Y. Sato, Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold, in: P. Lax, H. Fujita (Eds.), Nonlinear PDEs in Applied Sciences (US–Japan Seminar, Tokyo), North-Holland, Amsterdam, 1982, pp. 259–271.
[24] G. Segal, G. Wilson, Loop groups and equations of the KdV type, Publ. Math. IHES 61 (1985) 5–65.